

# Mathematical Induction

## Part Two

# Outline for Today

- ***Variations on Induction***
  - Starting later, taking different step sizes, and more!
- ***“Build Up” versus “Build Down”***
  - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
  - When one assumption isn't enough!

Recap from Last Time

Let  $P$  be some predicate. The **principle of mathematical induction** states that if

If it starts true...

$P(0)$  is true

...and it stays true...

and

$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...then it's always true.

**Theorem:** The sum of the first  $n$  powers of two is  $2^n - 1$ .

**Proof:** Let  $P(n)$  be the statement “the sum of the first  $n$  powers of two is  $2^n - 1$ .” We will prove, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ , from which the theorem follows.

For our base case, we need to show  $P(0)$  is true, meaning that the sum of the first zero powers of two is  $2^0 - 1$ . Since the sum of the first zero powers of two is zero and  $2^0 - 1$  is zero as well, we see that  $P(0)$  is true.

For the inductive step, assume that for some arbitrary  $k \in \mathbb{N}$  that  $P(k)$  holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \quad (1)$$

We need to show that  $P(k + 1)$  holds, meaning that the sum of the first  $k + 1$  powers of two is  $2^{k+1} - 1$ . To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore,  $P(k + 1)$  is true, completing the induction. ■

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New Stuff!

Variations on Induction: *Starting Later*

# Induction Starting at 0

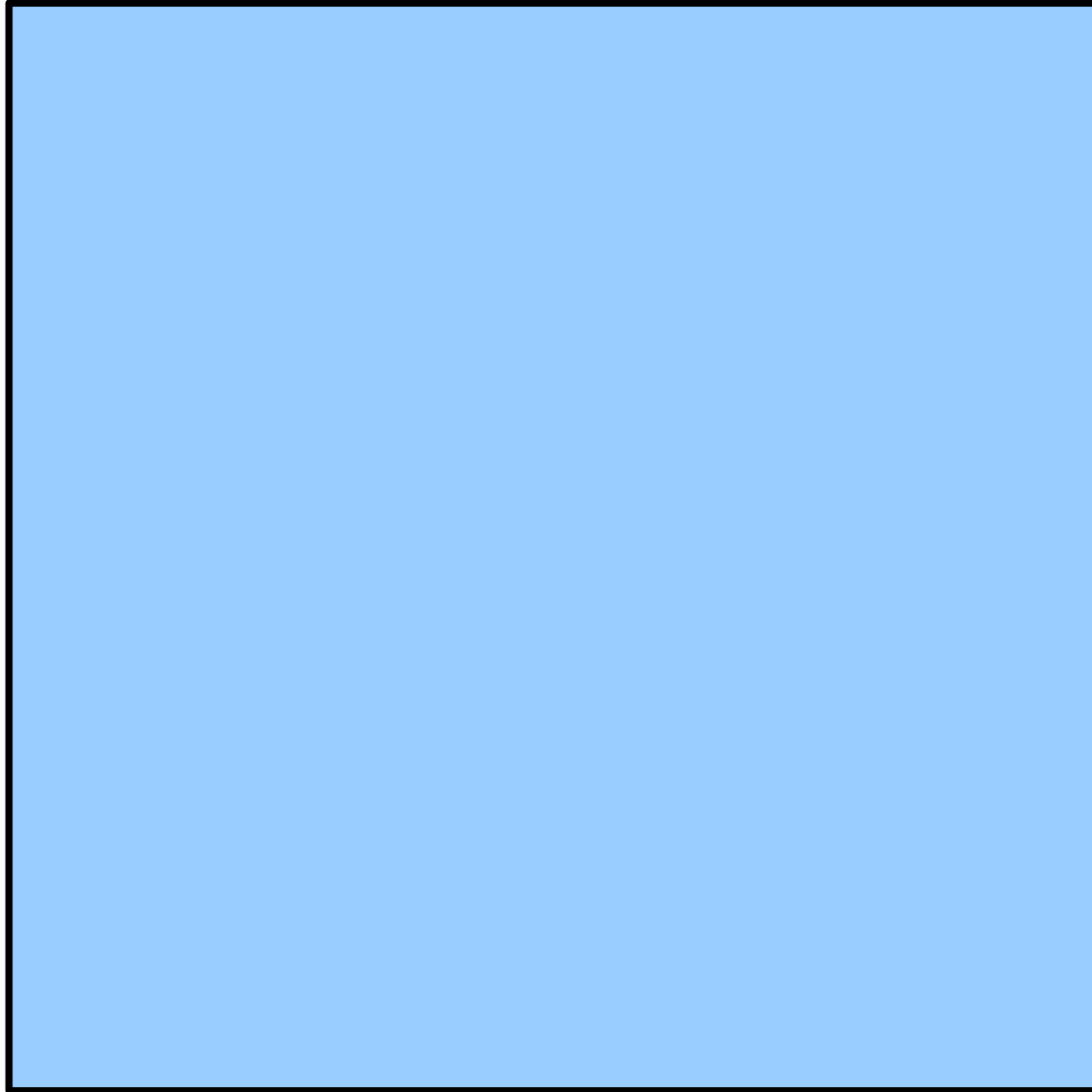
- To prove that  $P(n)$  is true for all natural numbers greater than or equal to 0:
  - Show that  $P(0)$  is true.
  - Show that for any  $k \geq 0$ , that if  $P(k)$  is true, then  $P(k+1)$  is true.
  - Conclude  $P(n)$  holds for all natural numbers greater than or equal to 0.

# Induction Starting at $m$

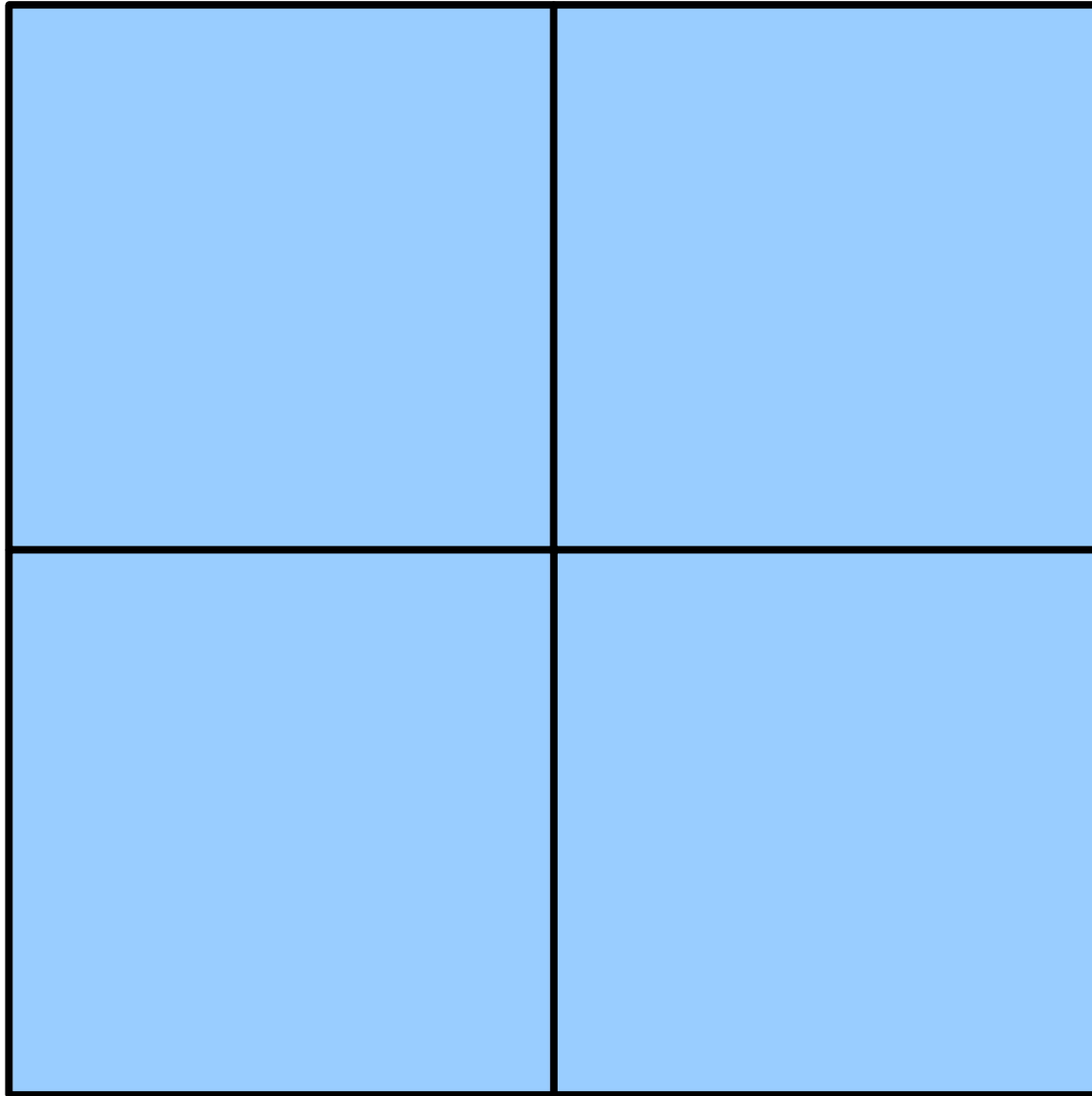
- To prove that  $P(n)$  is true for all natural numbers greater than or equal to  $m$ :
  - Show that  $P(m)$  is true.
  - Show that for any  $k \geq m$ , that if  $P(k)$  is true, then  $P(k+1)$  is true.
  - Conclude  $P(n)$  holds for all natural numbers greater than or equal to  $m$ .

Variations on Induction: ***Bigger Steps***

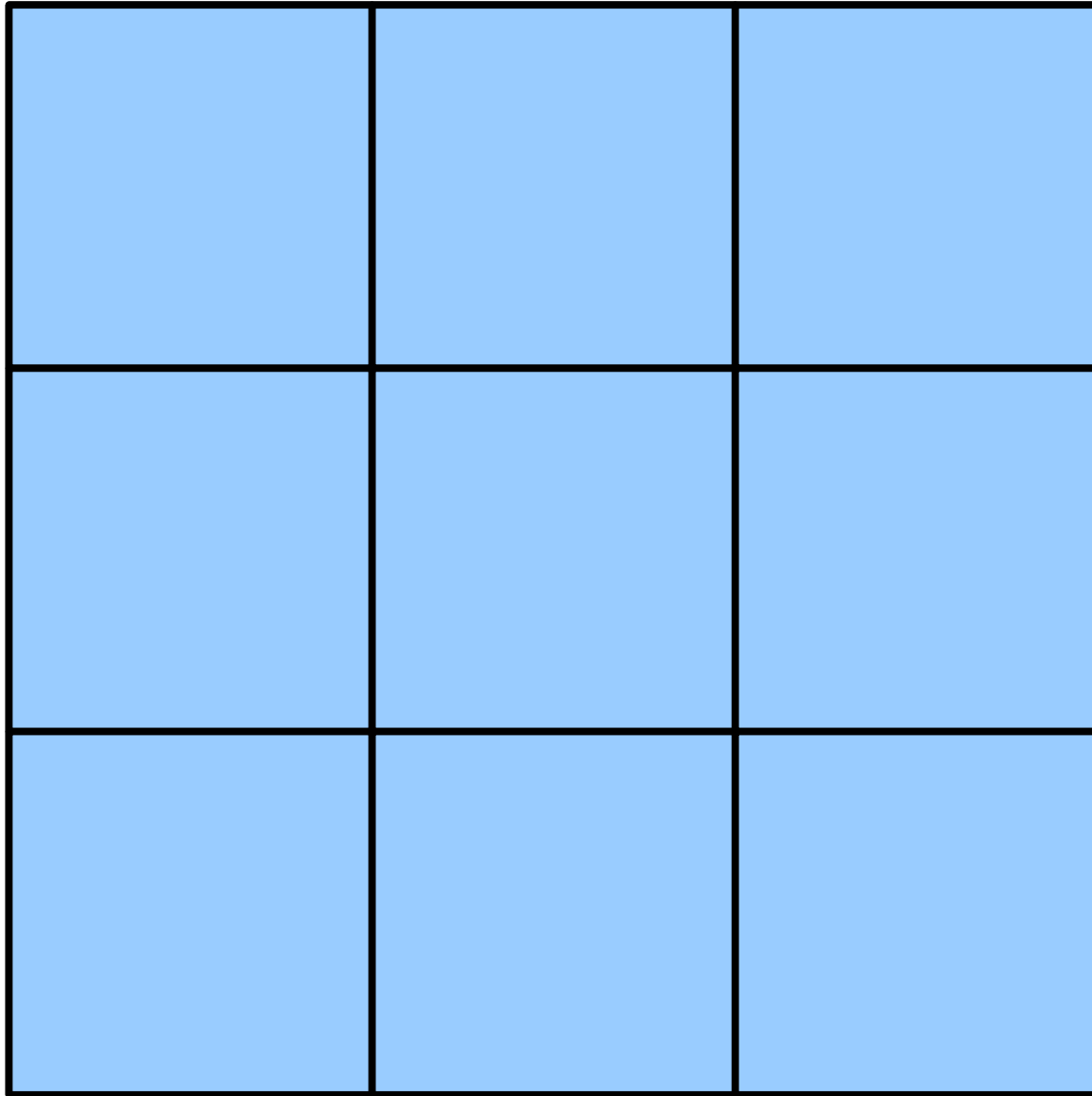
# Subdividing a Square



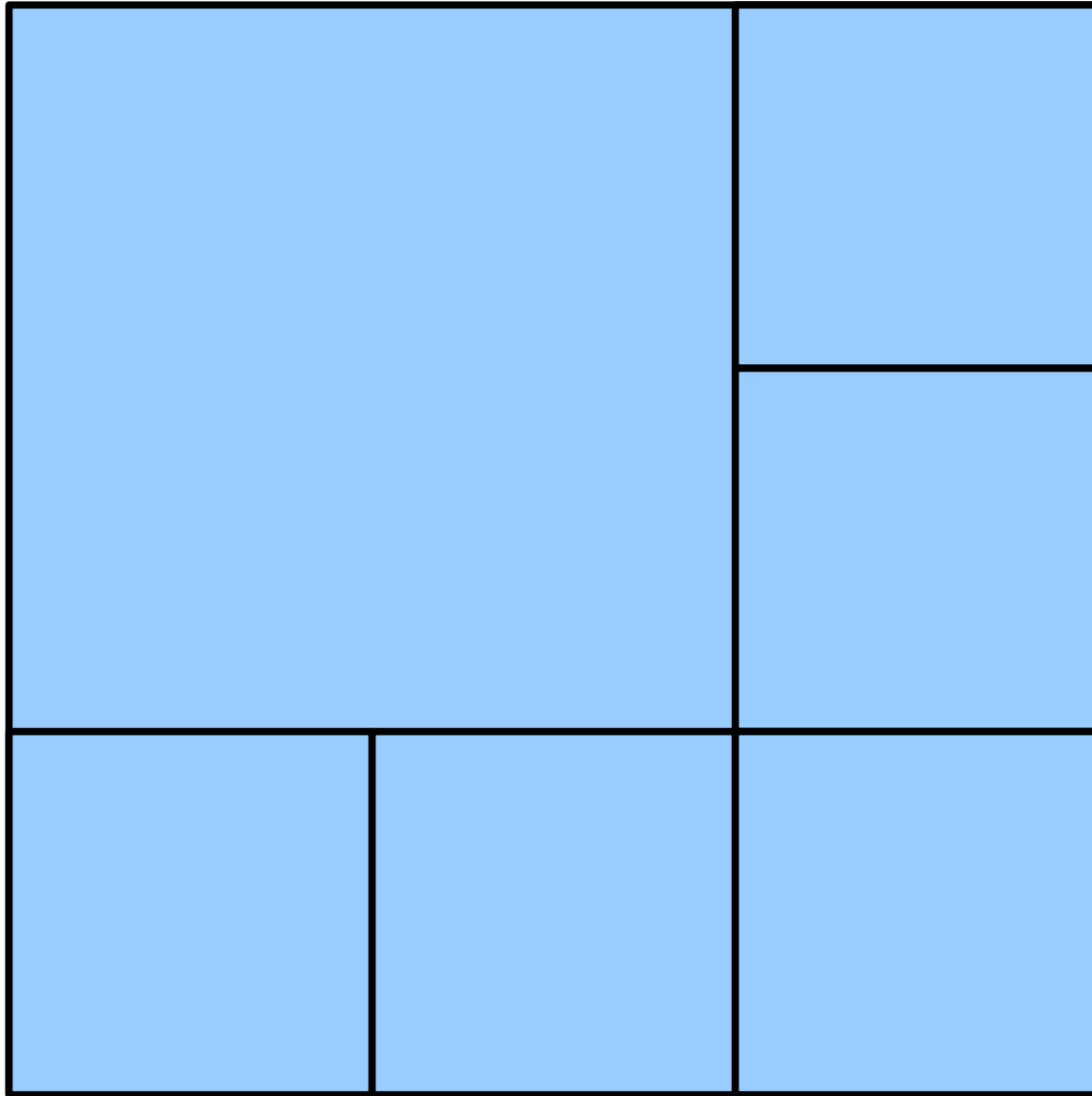
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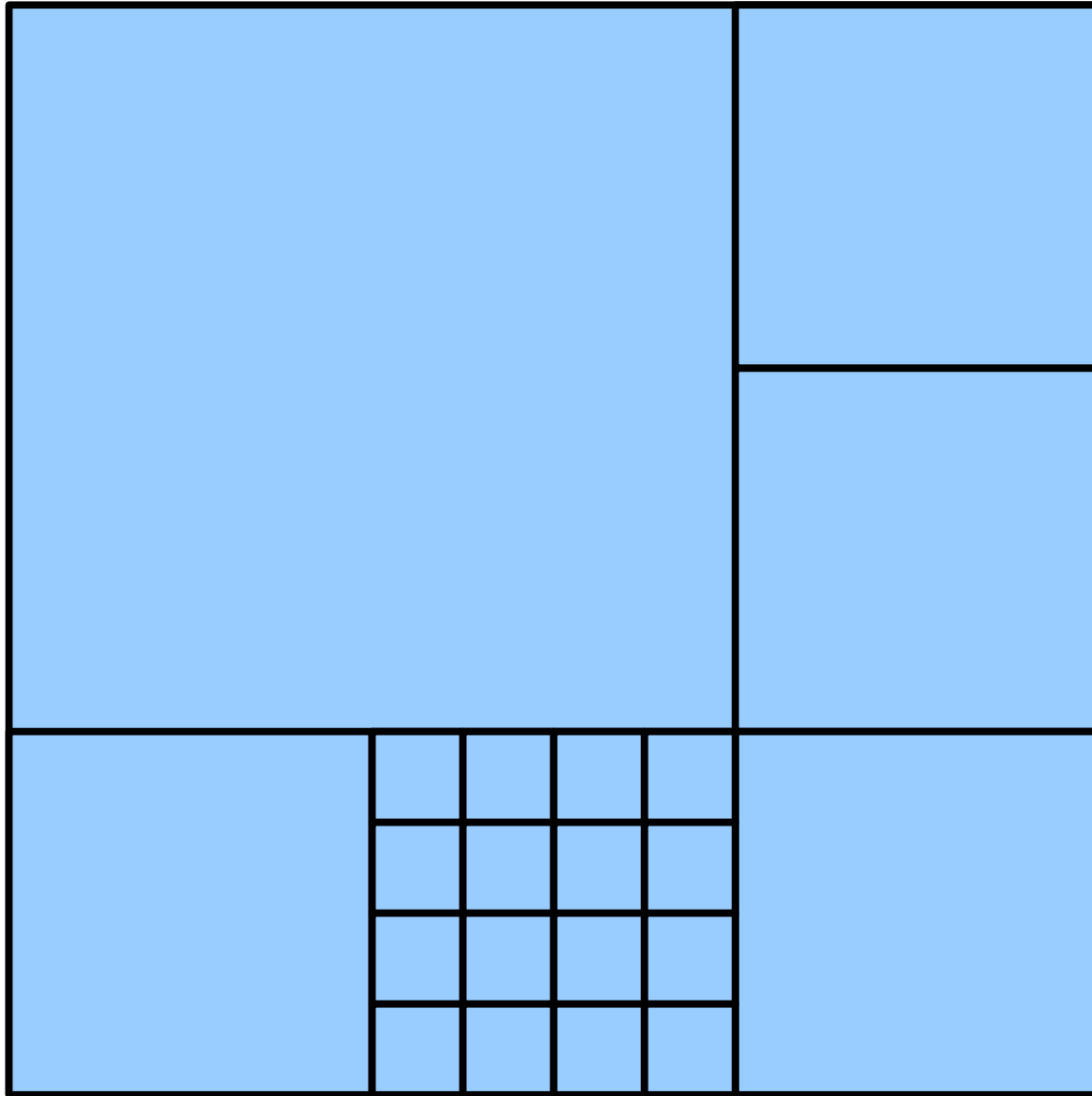
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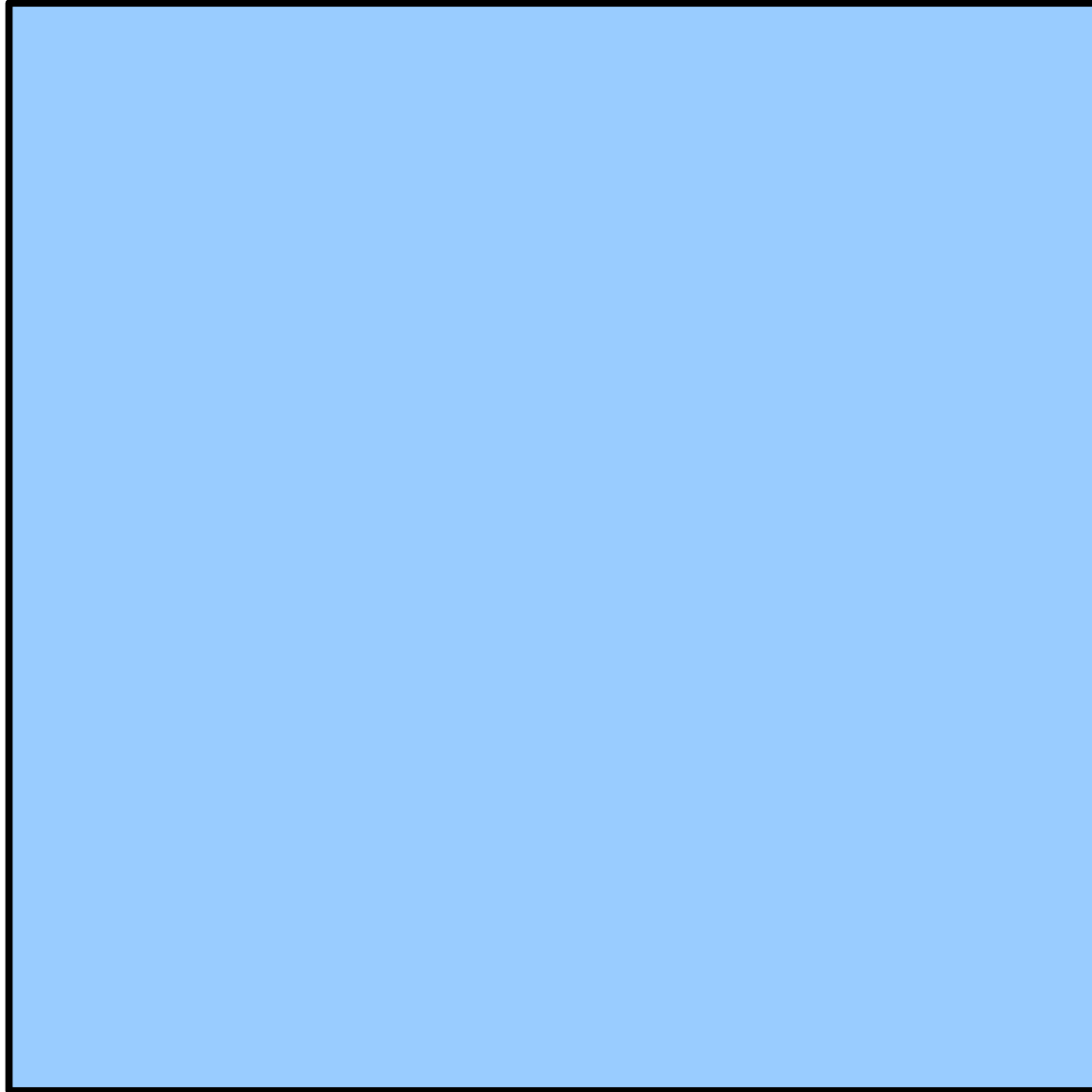
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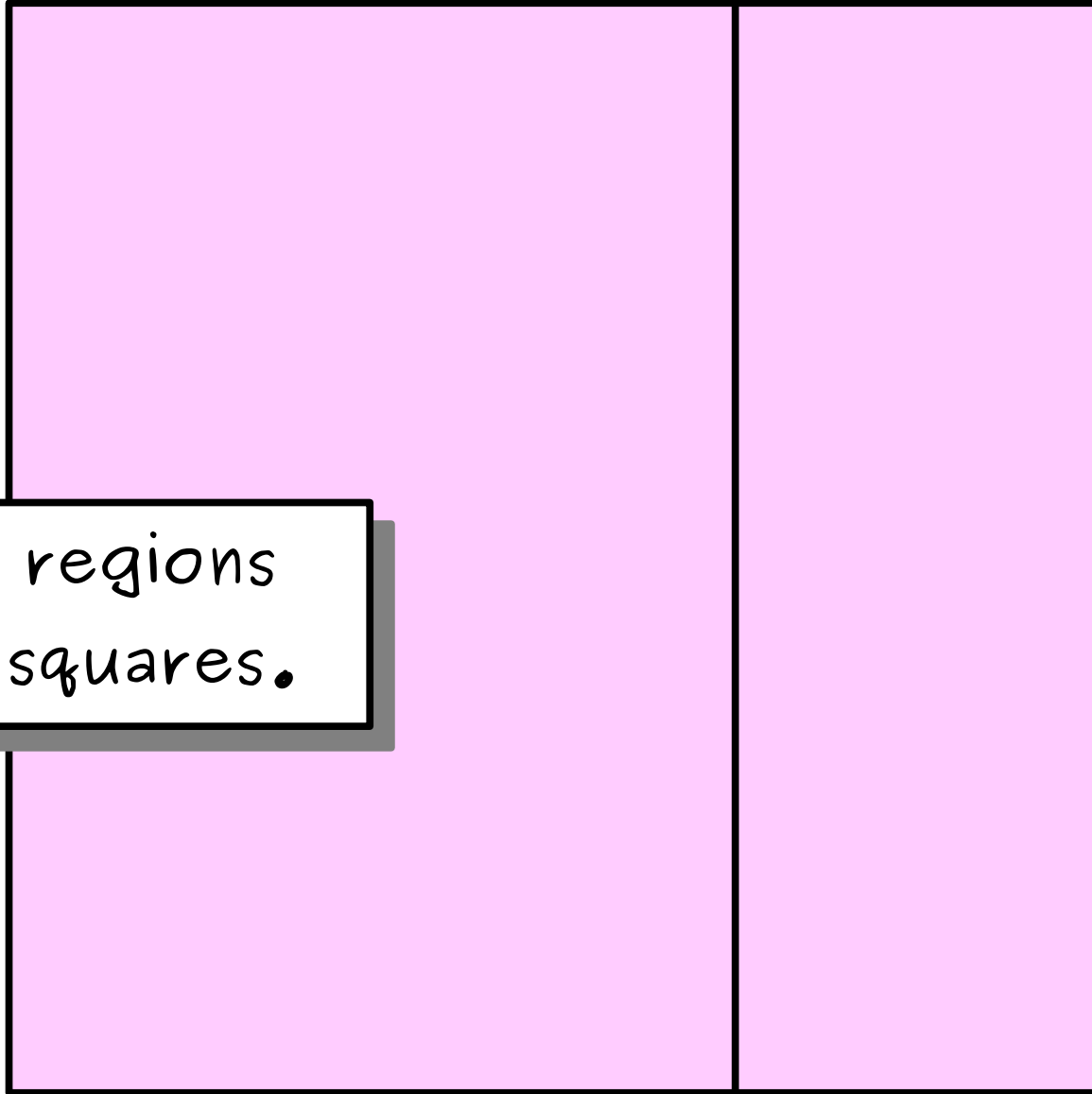
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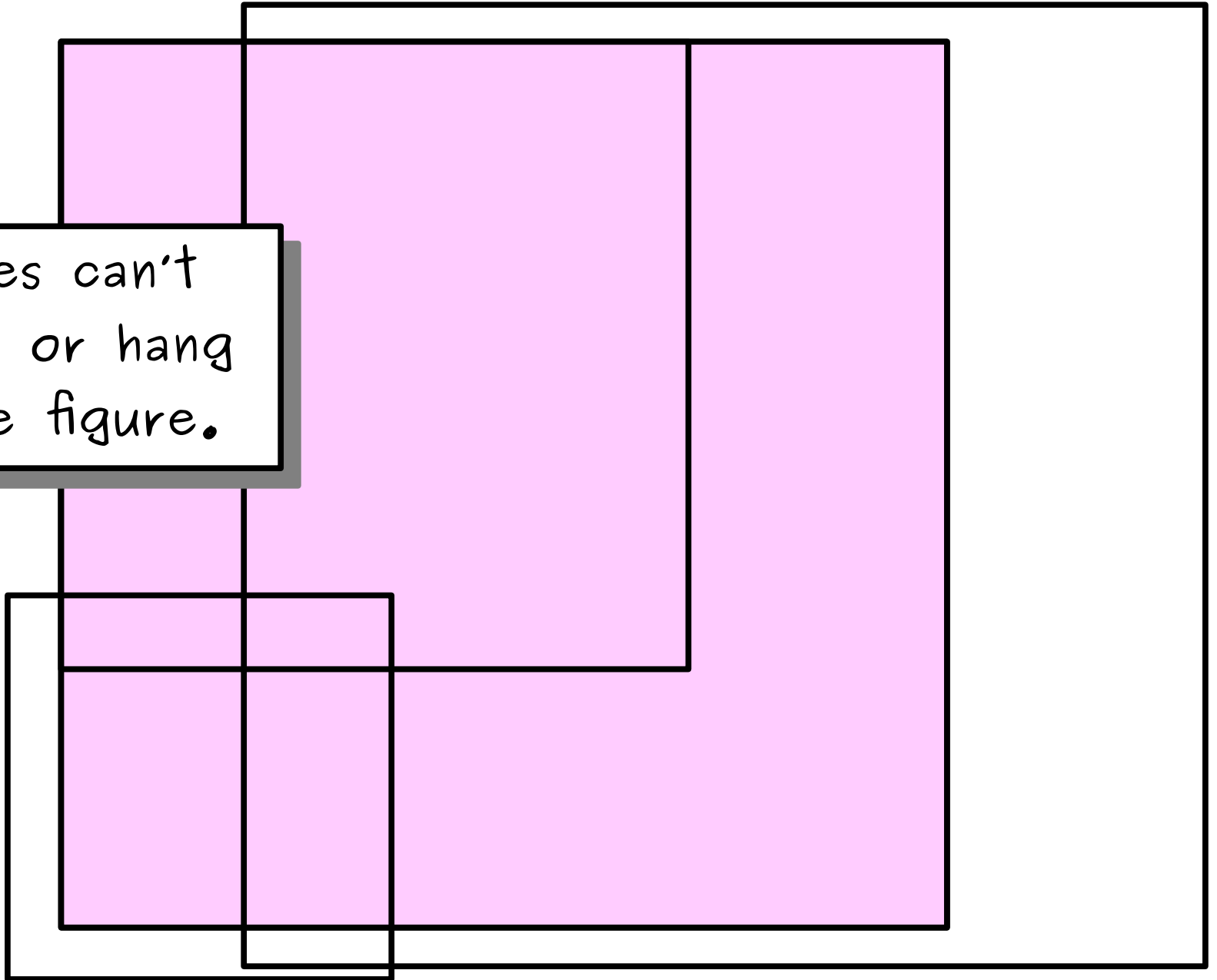
# Subdividing a Square



These regions  
aren't squares.

# Subdividing a Square

Squares can't  
overlap or hang  
off the figure.



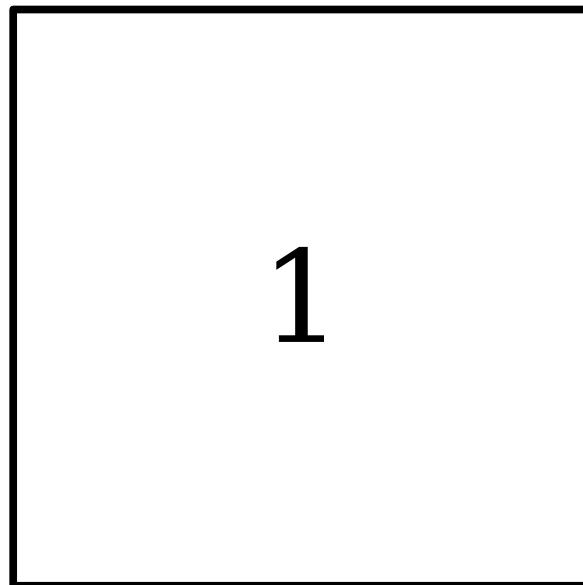
For what values of  $n$  can a square be subdivided into  $n$  squares?

1 2 3 4 5 6 7 8 9 10 11 12

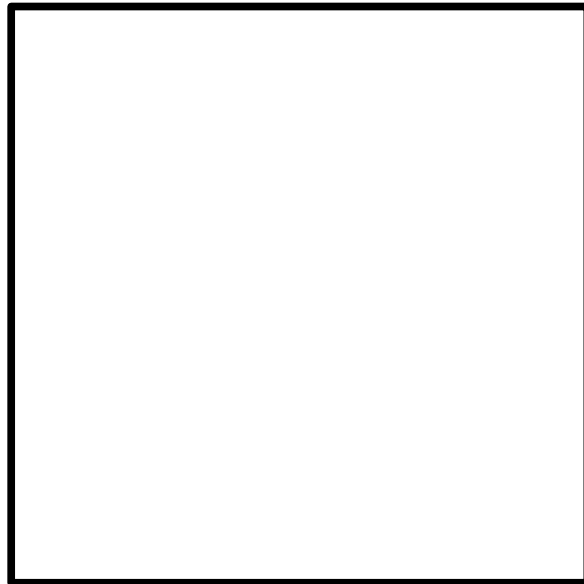
Give it a try! Enter your guess as a list of values.

***Respond at [pollev.com/cs103](https://pollev.com/cs103)***

1 2 3 4 5 6 7 8 9 10 11 12

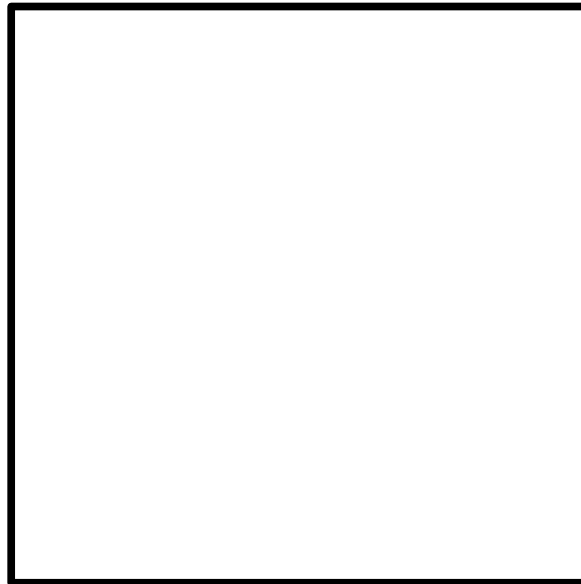


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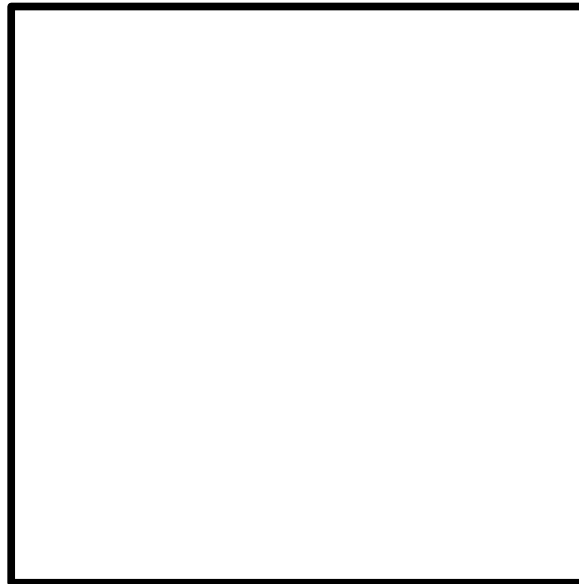
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Each of the original corners needs to be covered by a corner of the new smaller squares.



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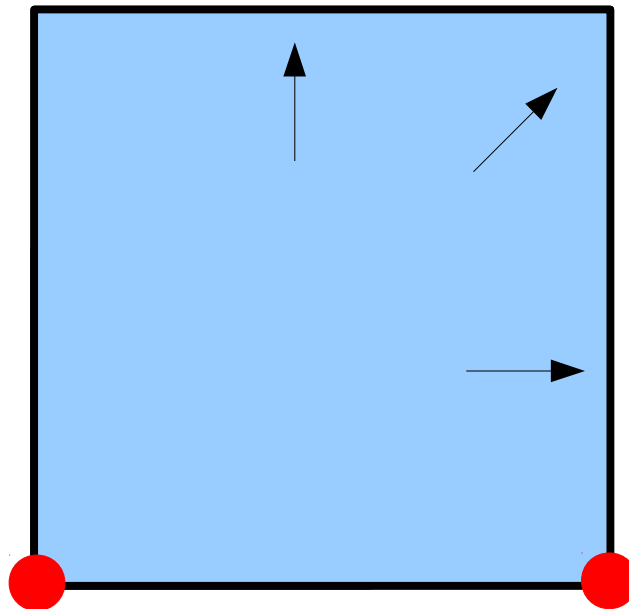


# corners: 4

# squares:  $<4$

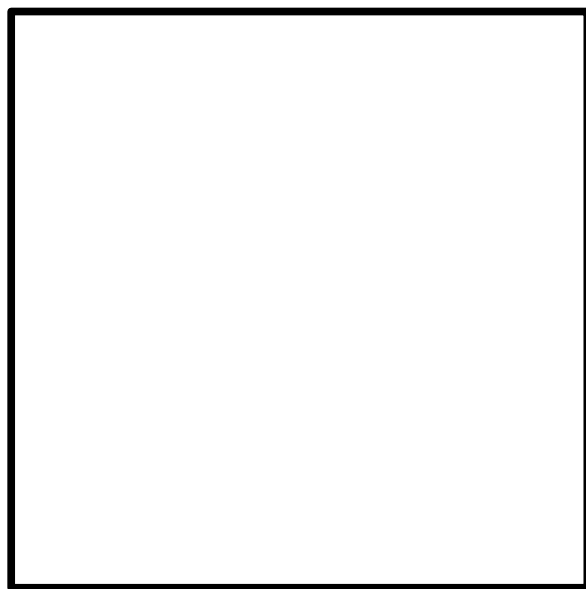
1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

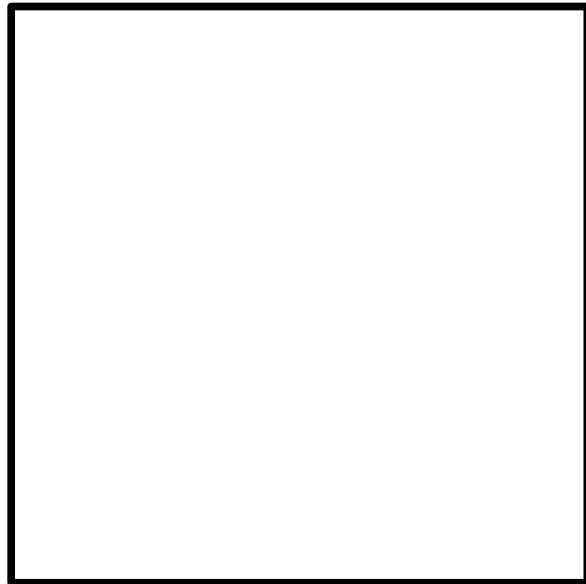
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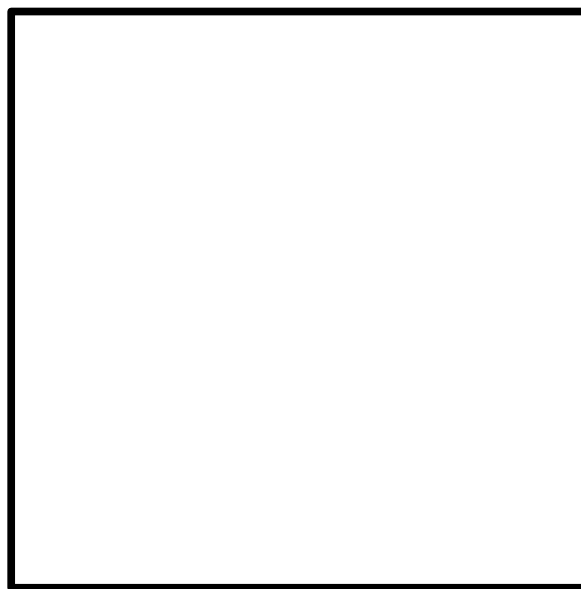
1 2 3 4 5 6 7 8 9 10 11 12

1	2
4	3

1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12

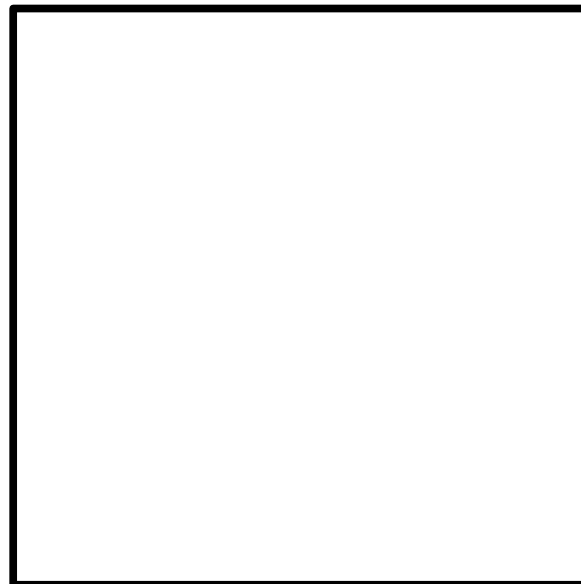


# corners: 4

# squares: 5

1 2 3 4 5 6 7 8 9 10 11 12

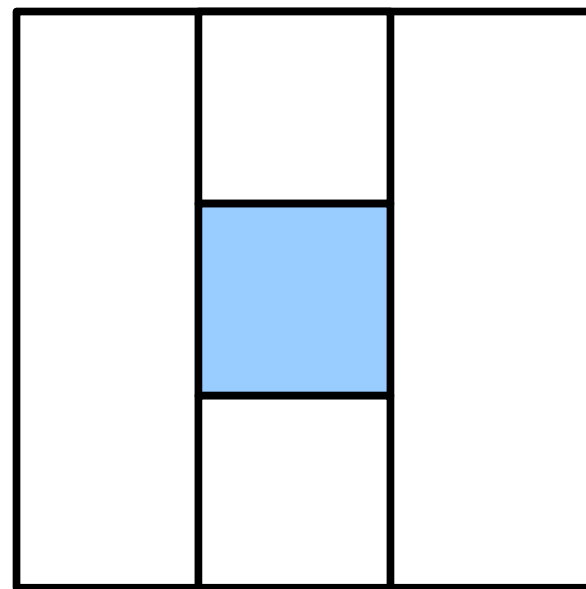
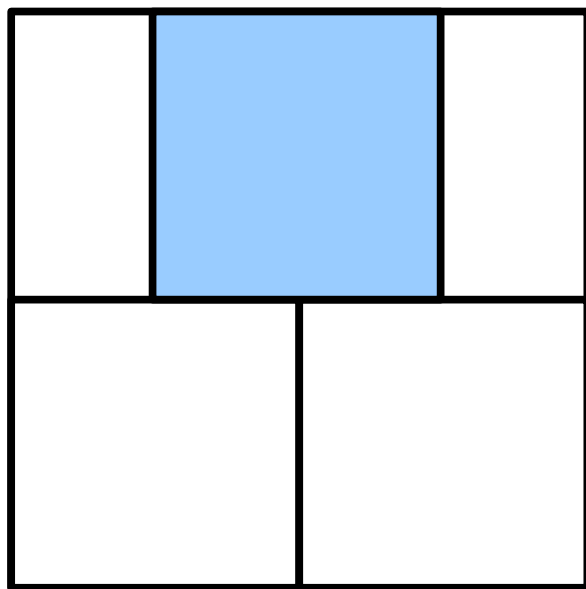
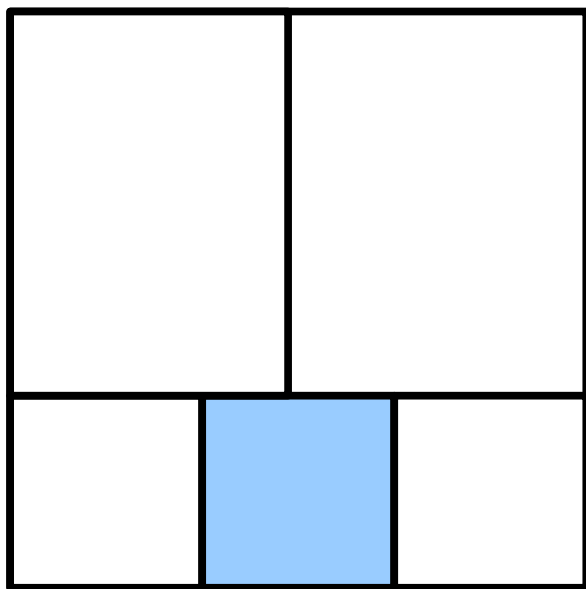
At least one square  
cannot be covering  
*any* of the original  
corners



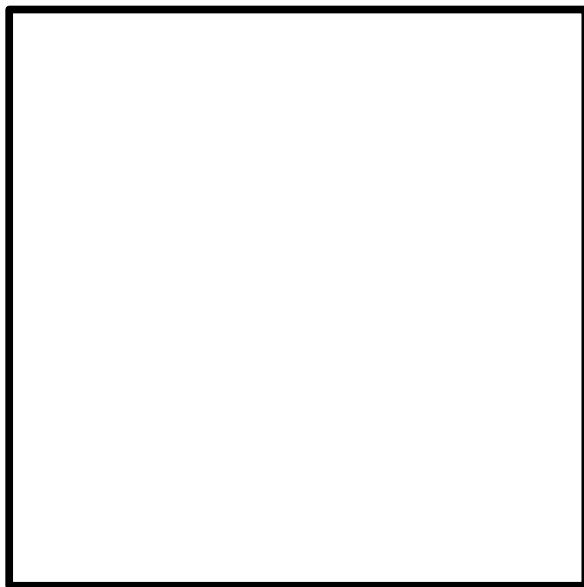
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1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12



1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1		2
		3
6	5	4

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

5	6	1
4	7	
3		2

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1			
2	8		
3			
4	5	6	7

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 ~~9~~ 10 11 12

1	2	3
8	9	4
7	6	5

1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	2	3	
8	9	3	
7		10	4
		6	5

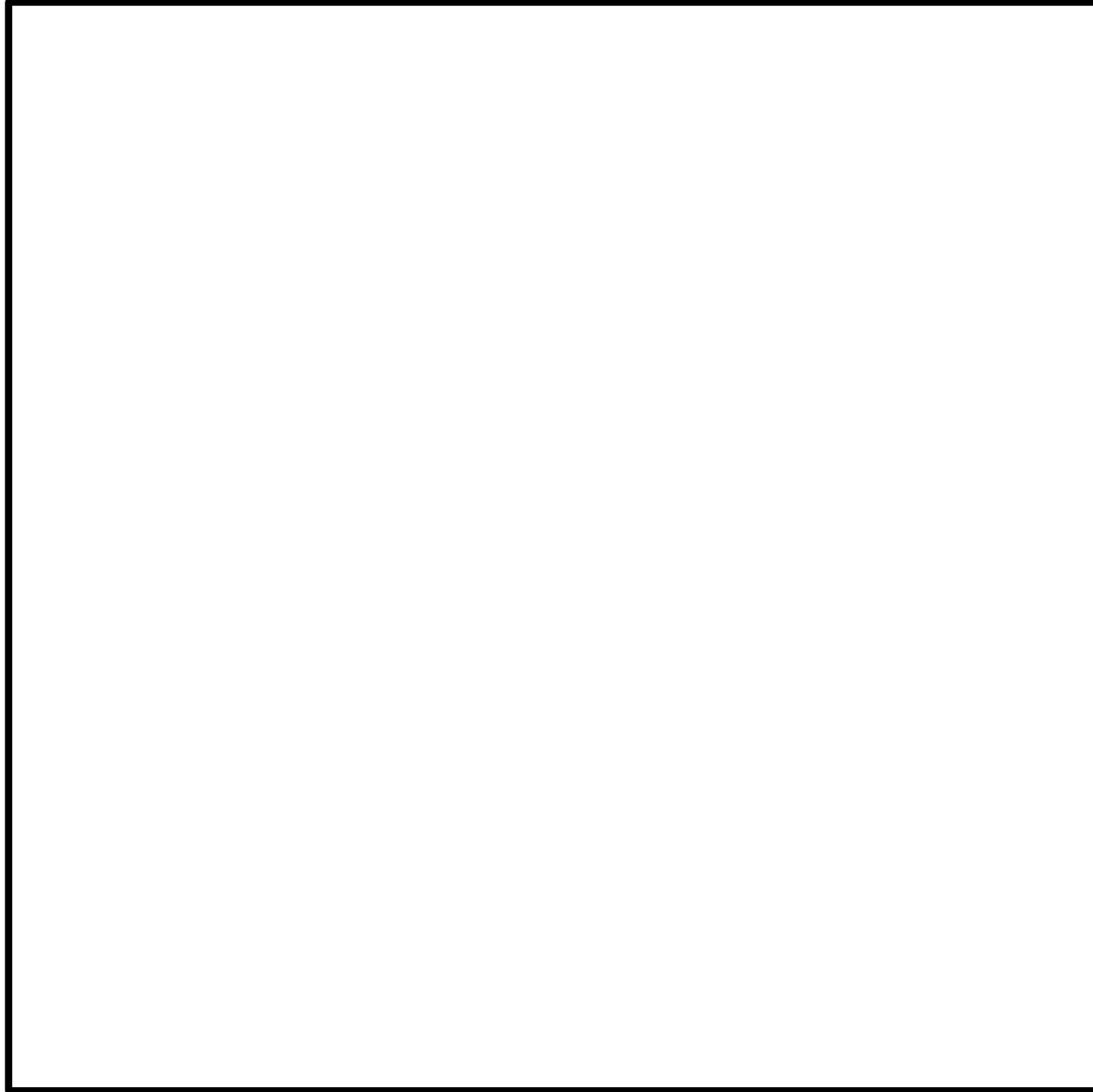
1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	10		9
2	11		8
3	5	6	7
4			

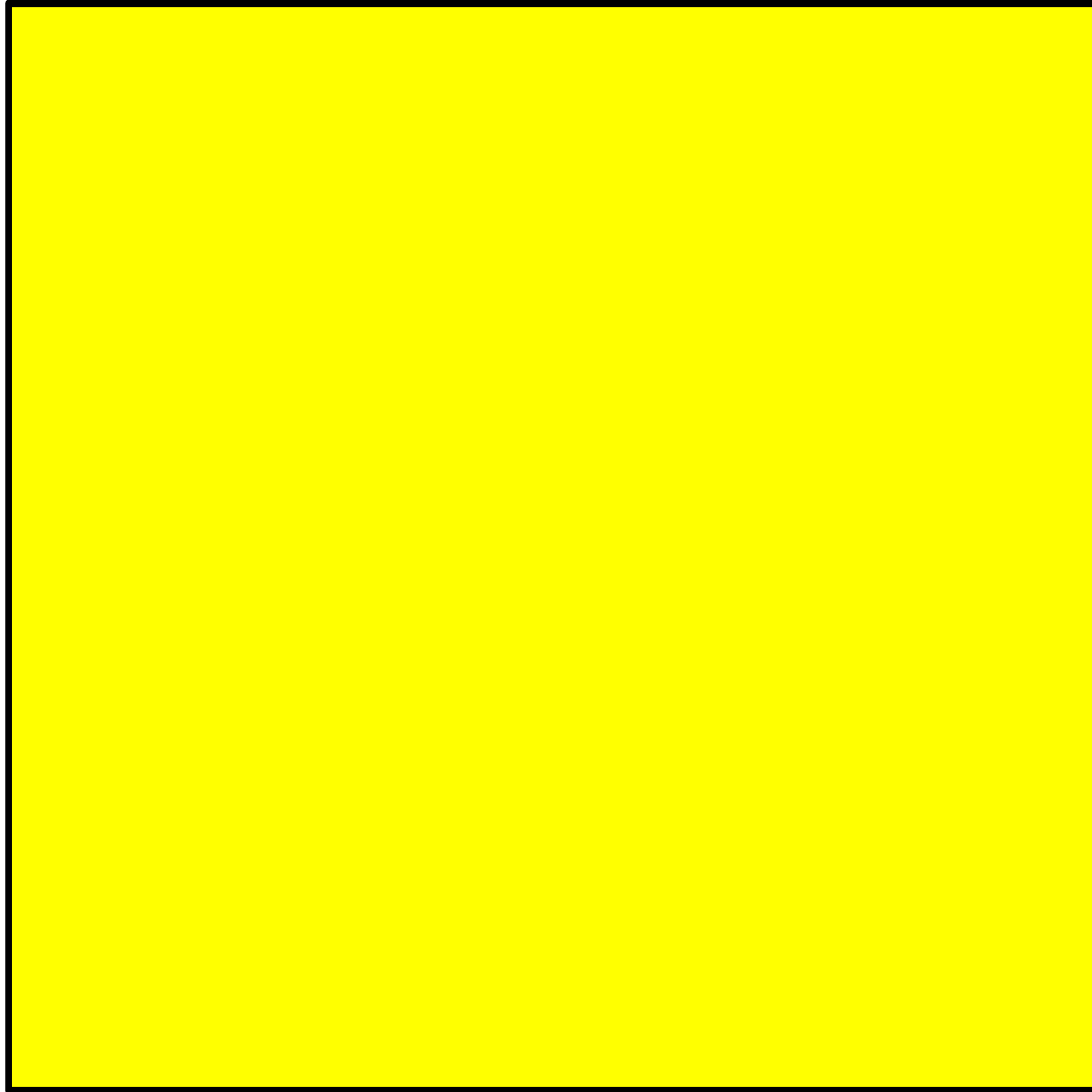
1   ~~2~~   ~~3~~   4   ~~5~~   6   7   8   9   10   11   12

1	2	3	
8	9	10	4
	12	11	
7	6	5	

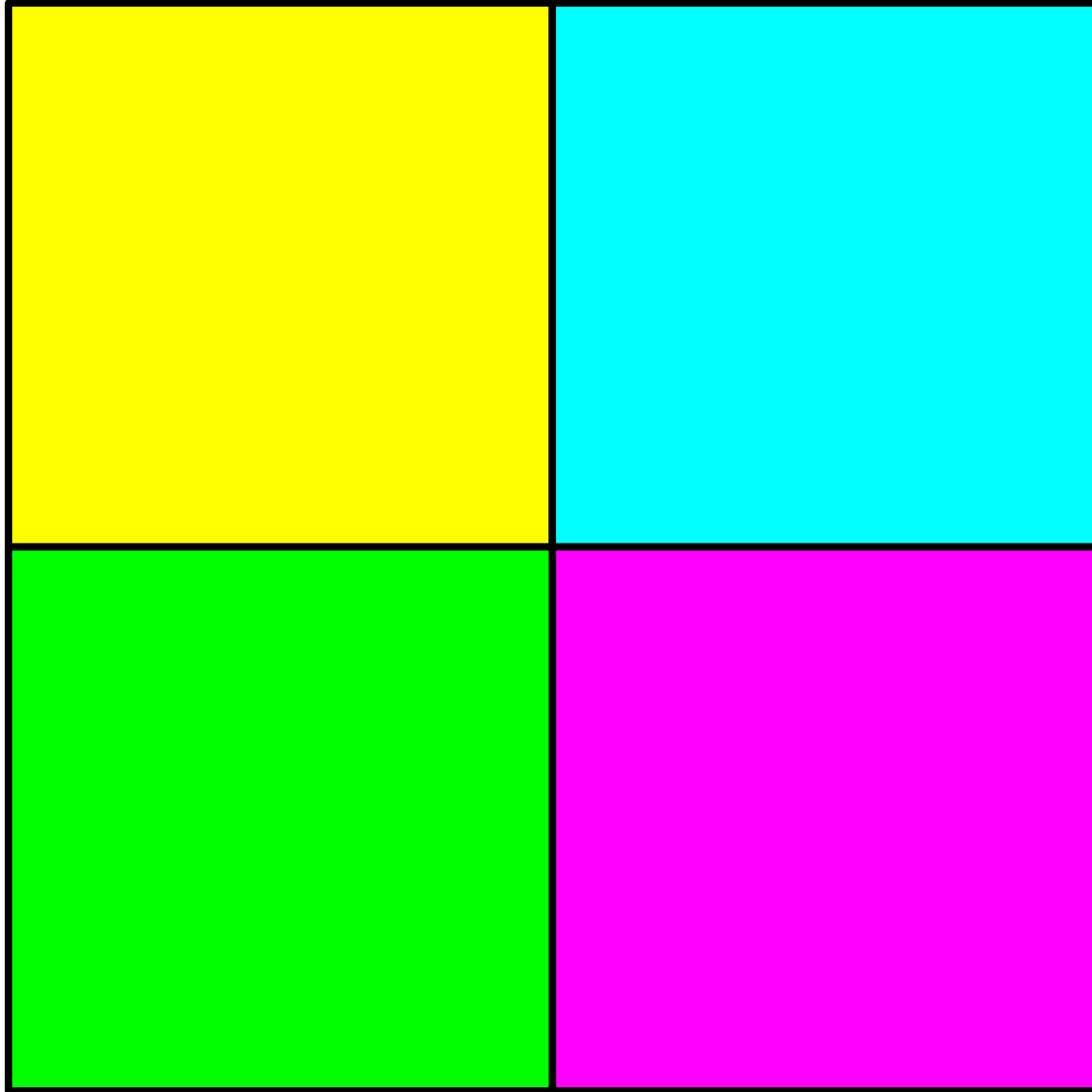
# An Insight



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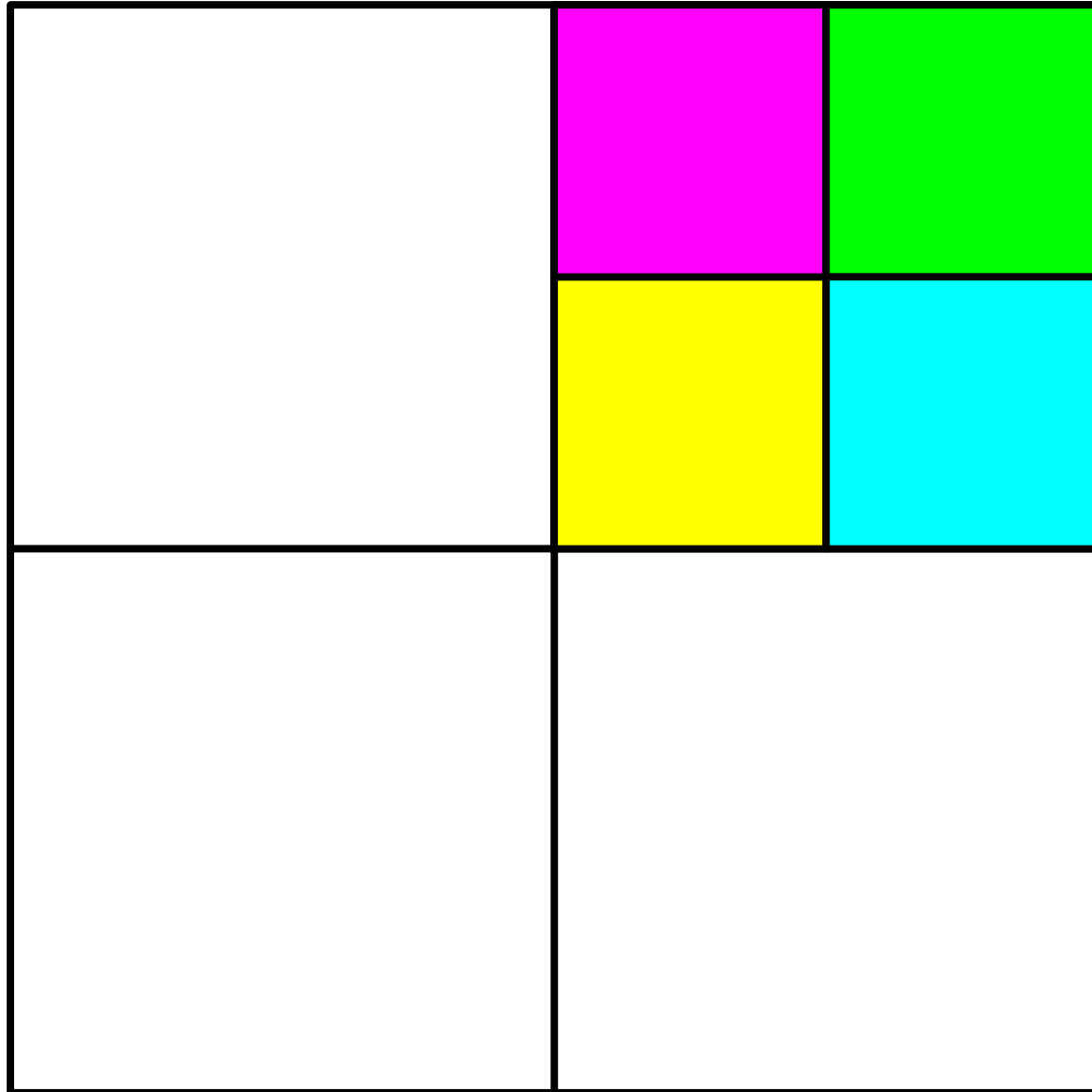
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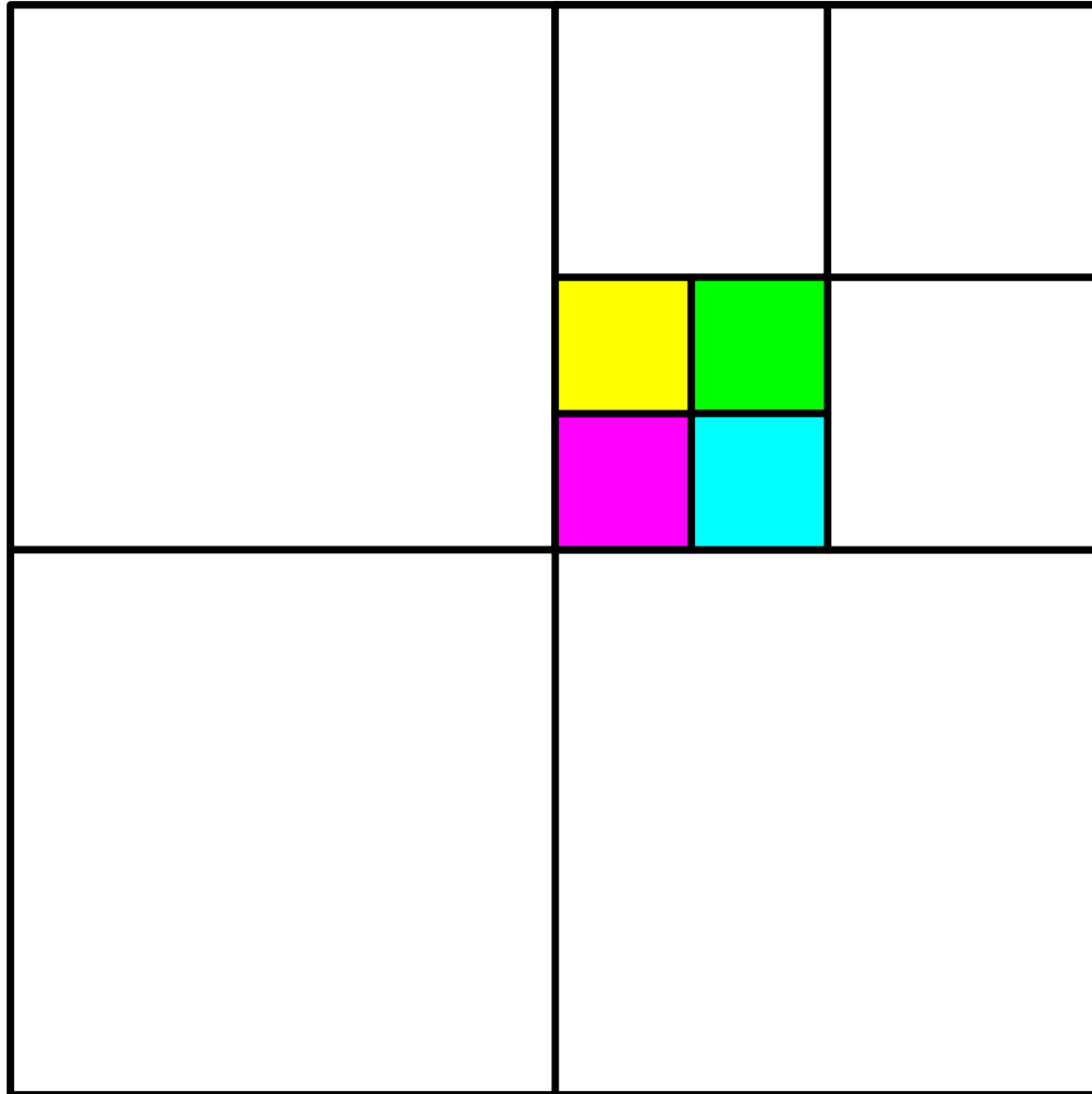

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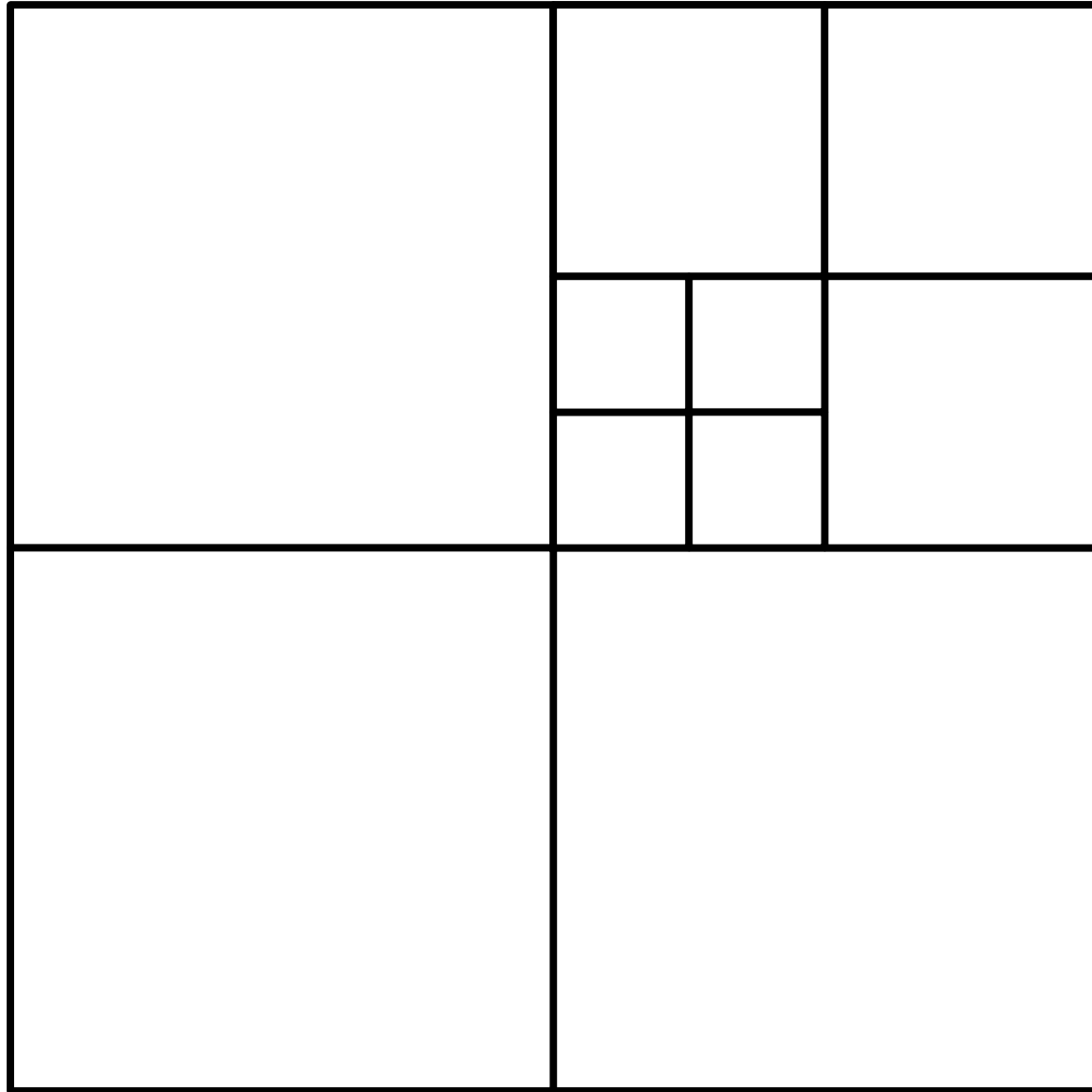
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# An Insight

- If we can subdivide a square into  $n$  squares, we can also subdivide it into  $n + 3$  squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into  $n$  squares for any  $n \geq 6$ :
  - For multiples of three, start with 6 and keep adding three squares until  $n$  is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until  $n$  is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until  $n$  is reached.

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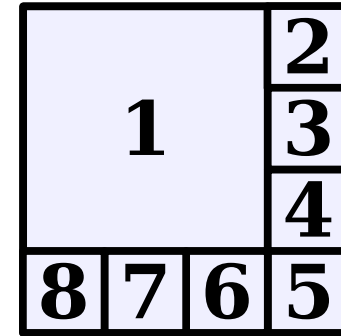
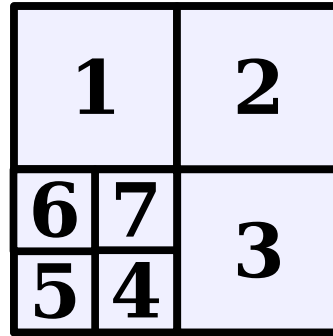
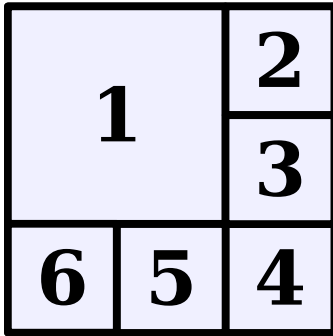
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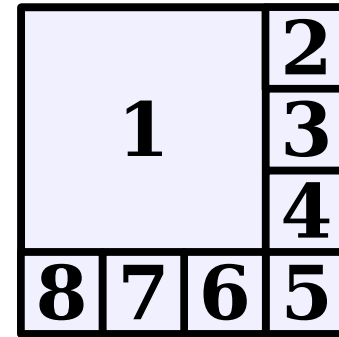
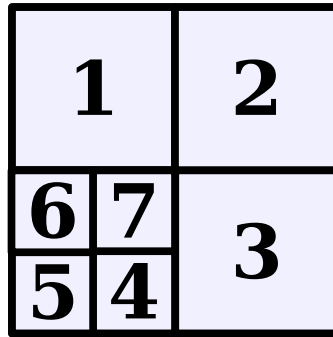
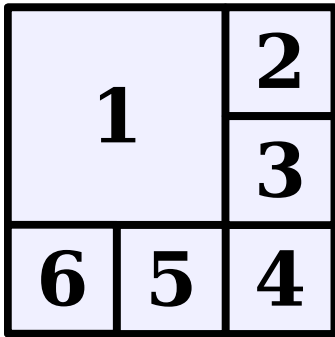
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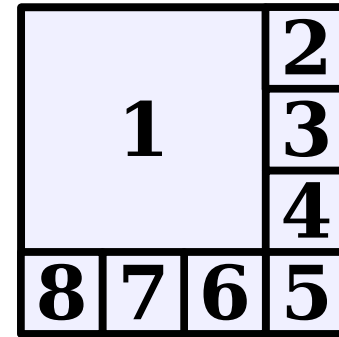
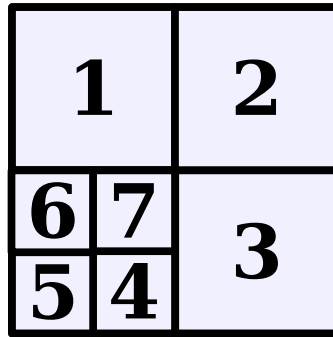
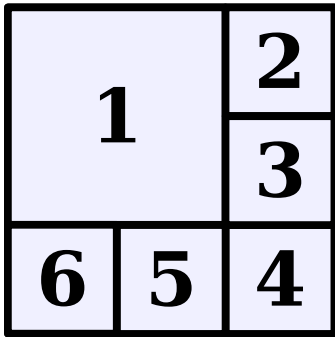


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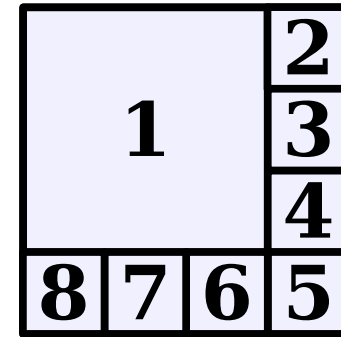
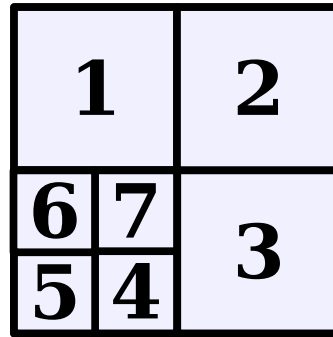
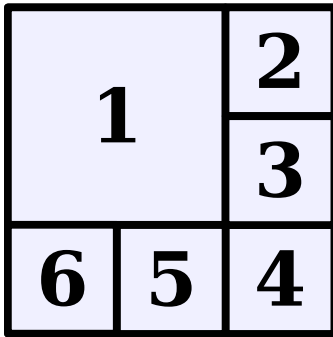


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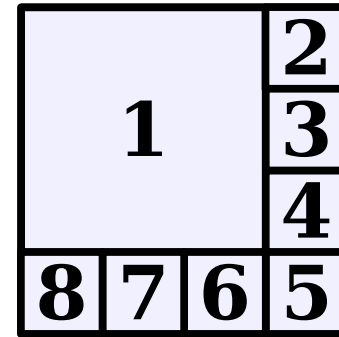
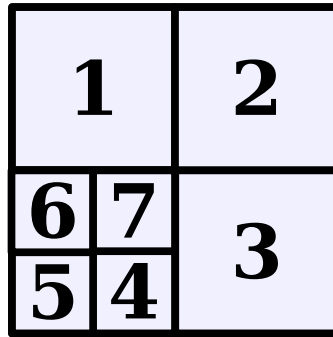
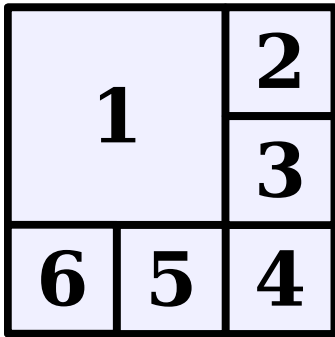


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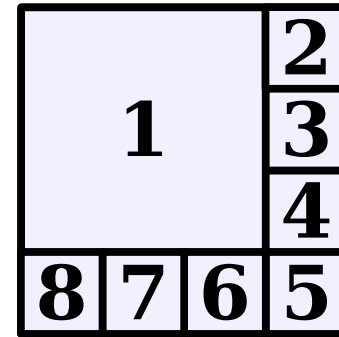
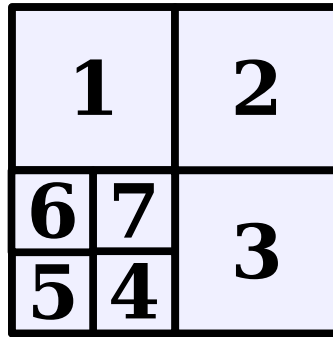
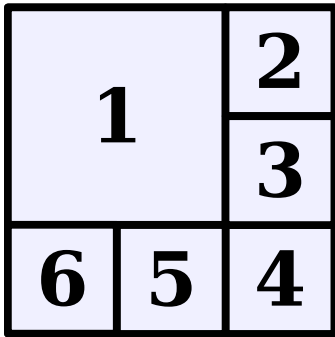


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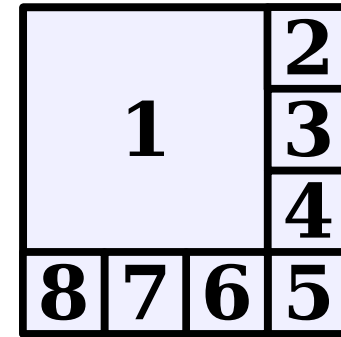
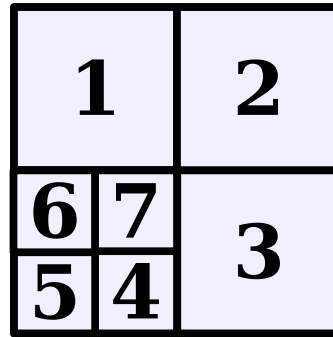
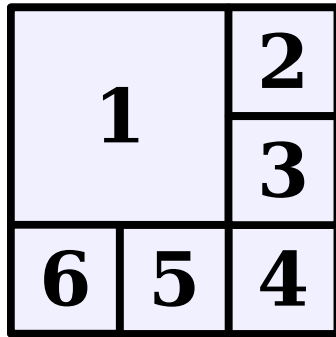


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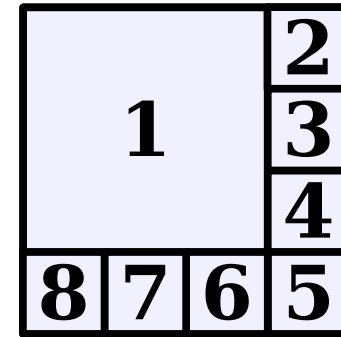
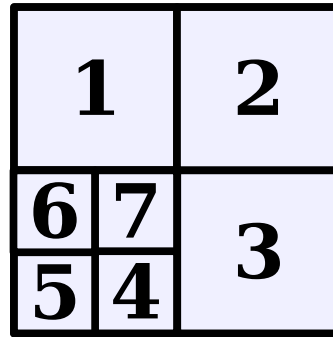
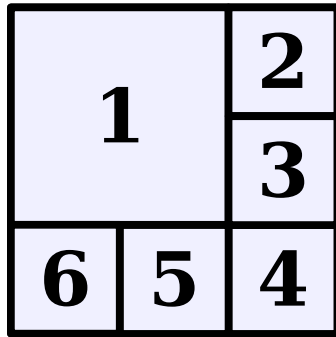


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# Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that...
  - ... you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - ... you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

# More on Square Subdivisions

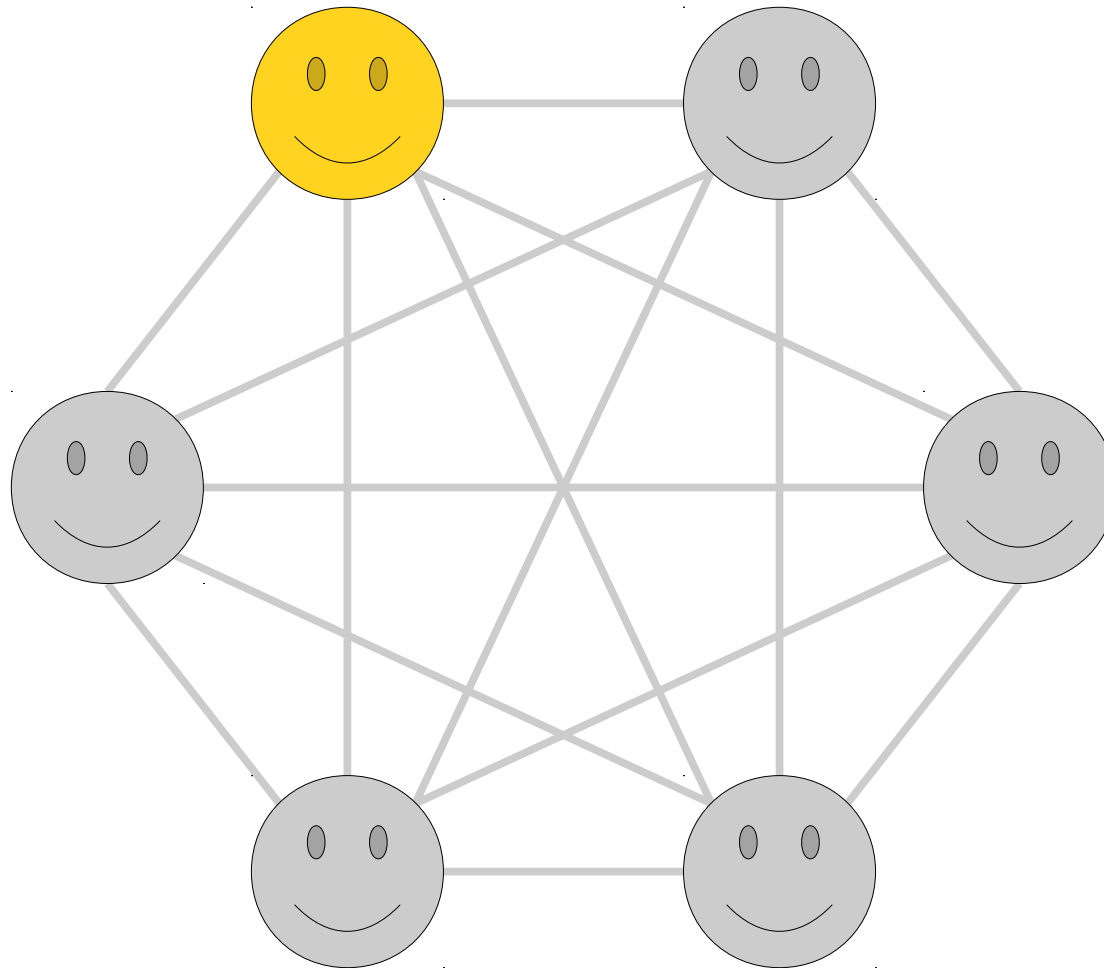
- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on [\*Squaring the Square\*](#).

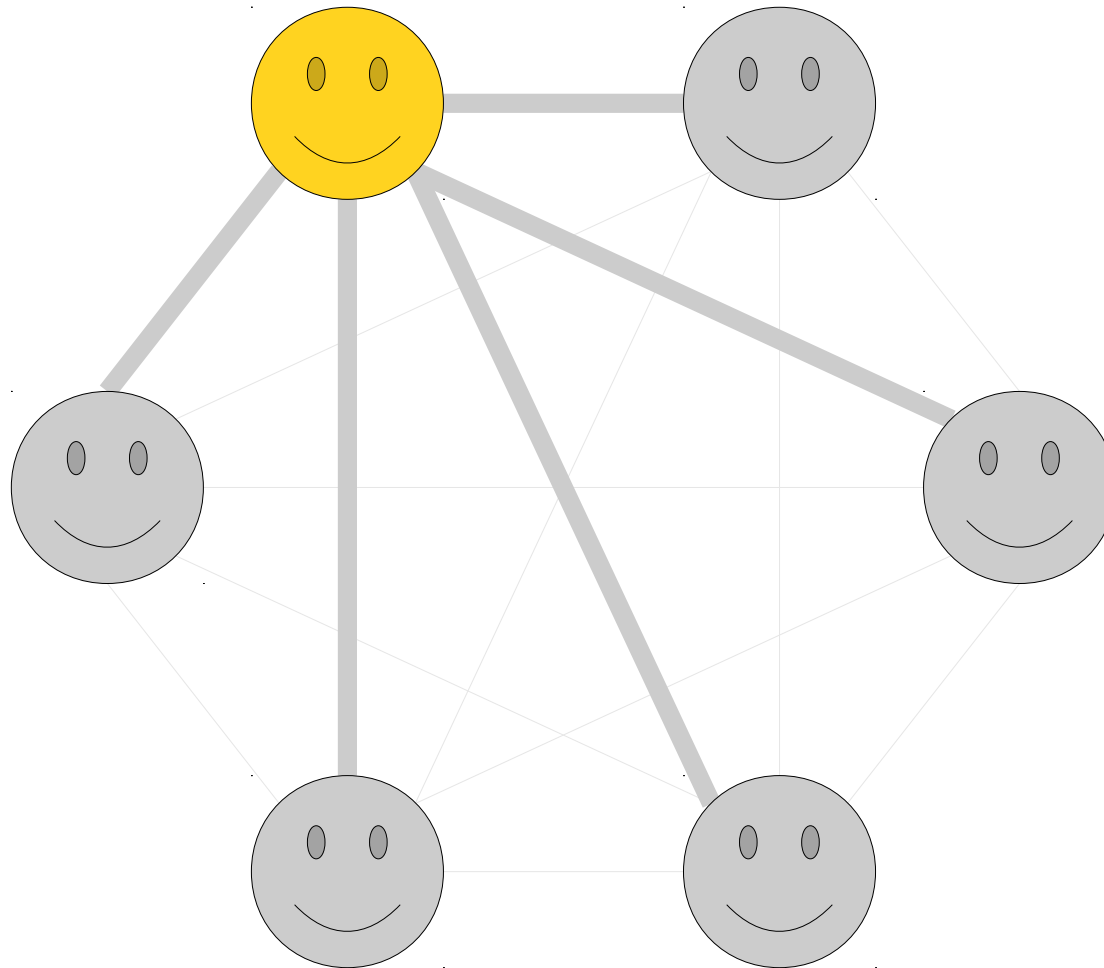
# Ramsey Revisited

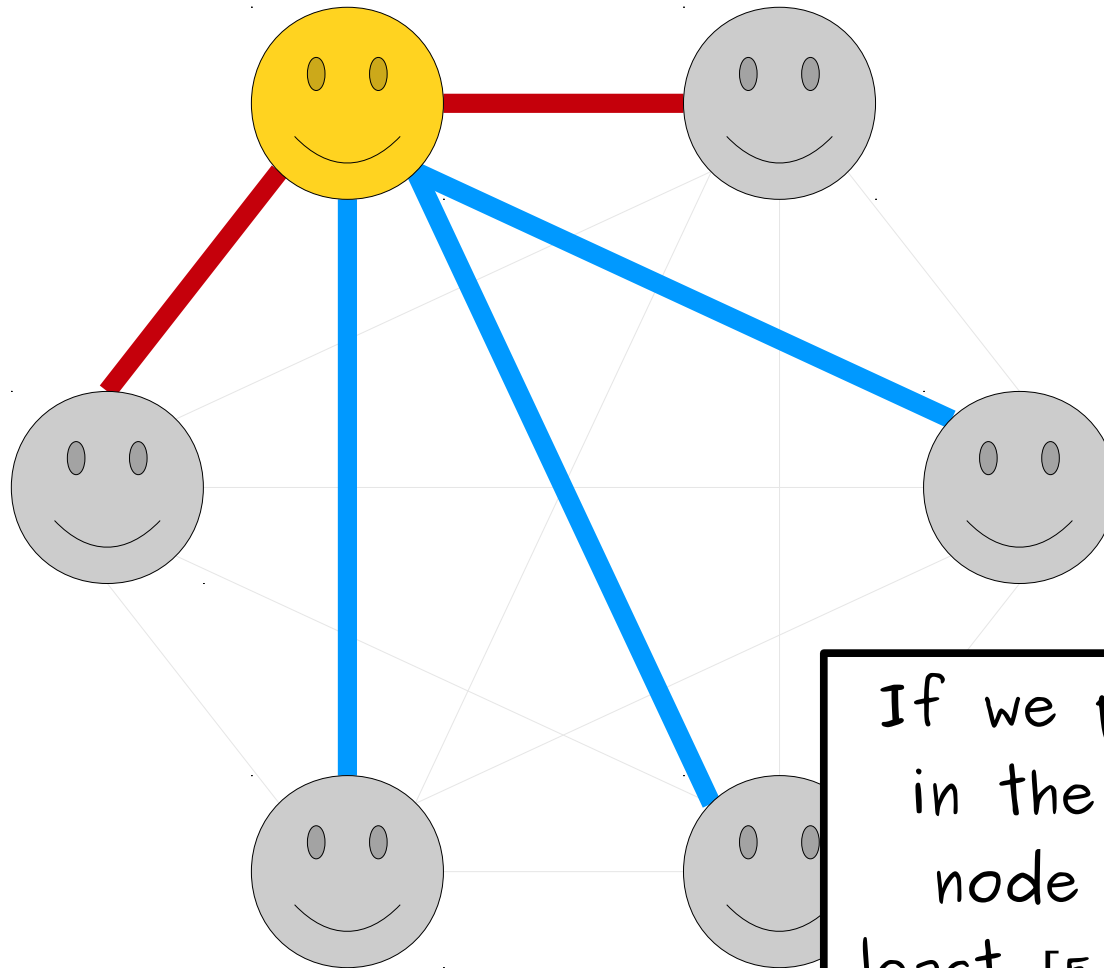
# Ramsey Revisited

- In lecture, we proved the Theorem on Friends and Strangers: any 6-clique whose edges are painted one of two colors contains a monochrome triangle.
- On PS4, you're proving that any 17-clique whose edges are painted one of three colors has a monochrome triangle.
- What about if you use four colors? Five colors? Six colors?

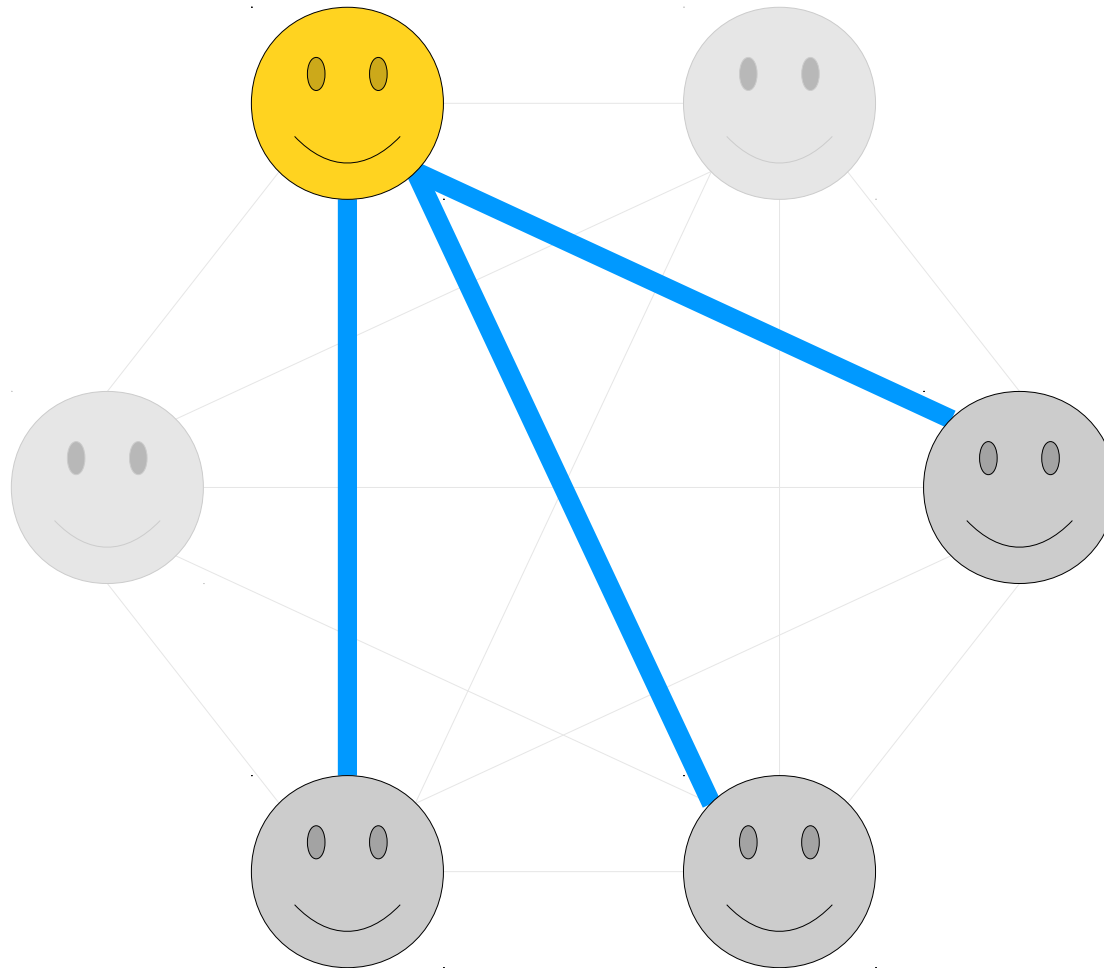
# Refresher on Friends and Strangers

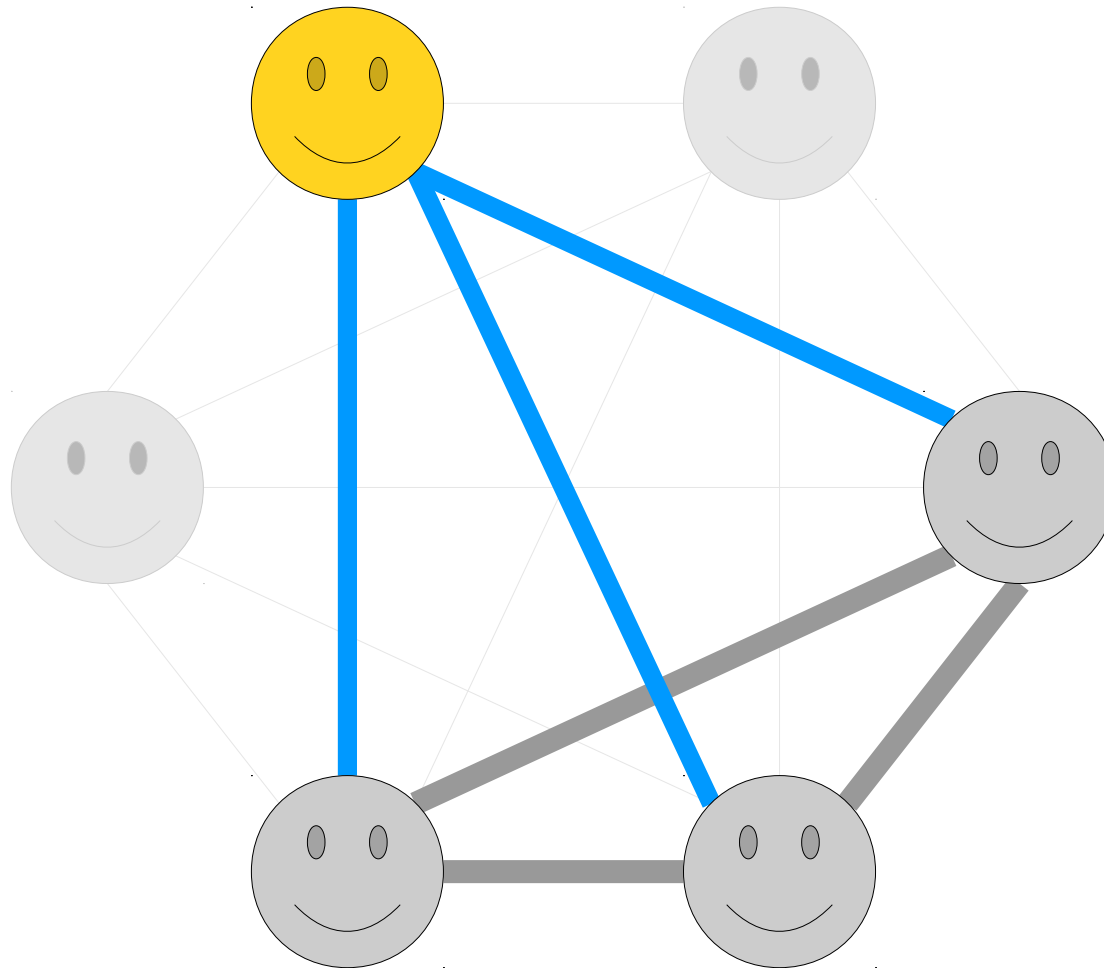


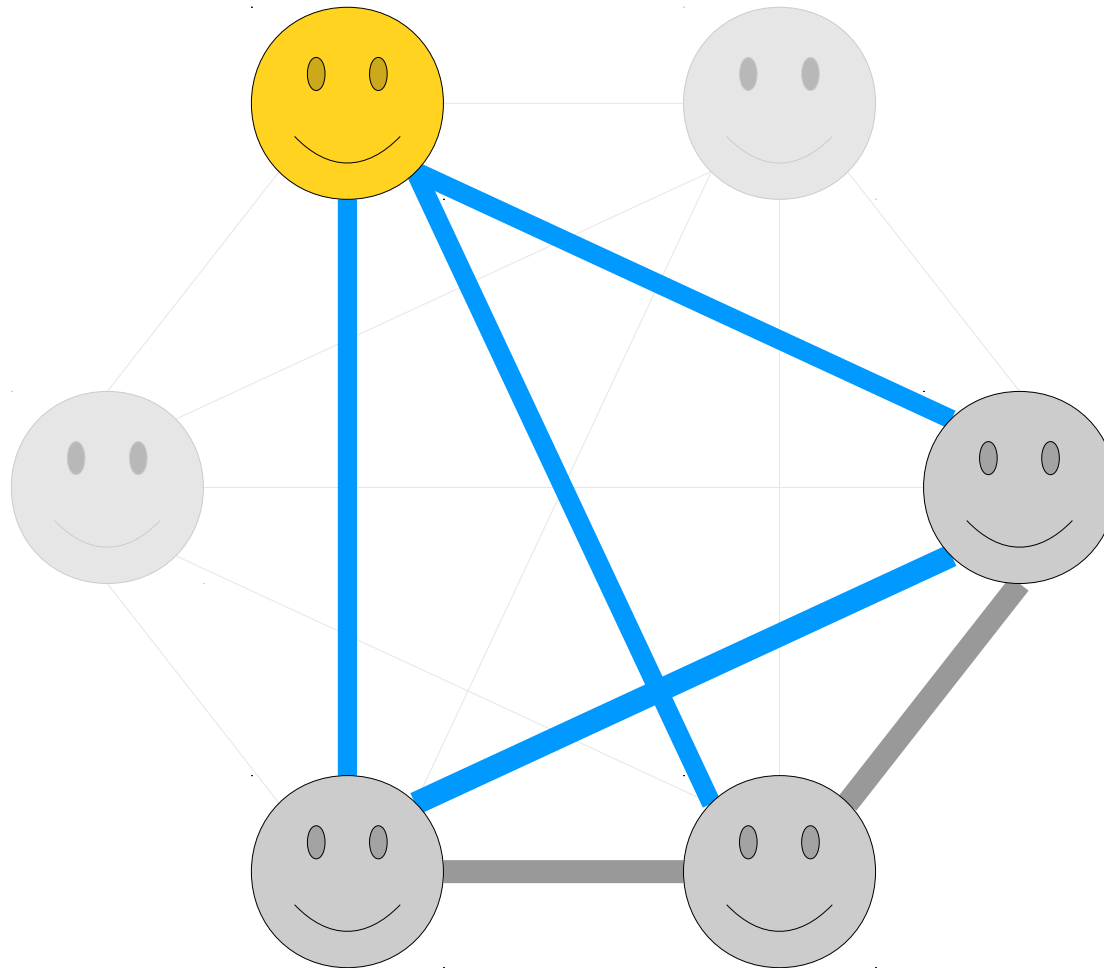


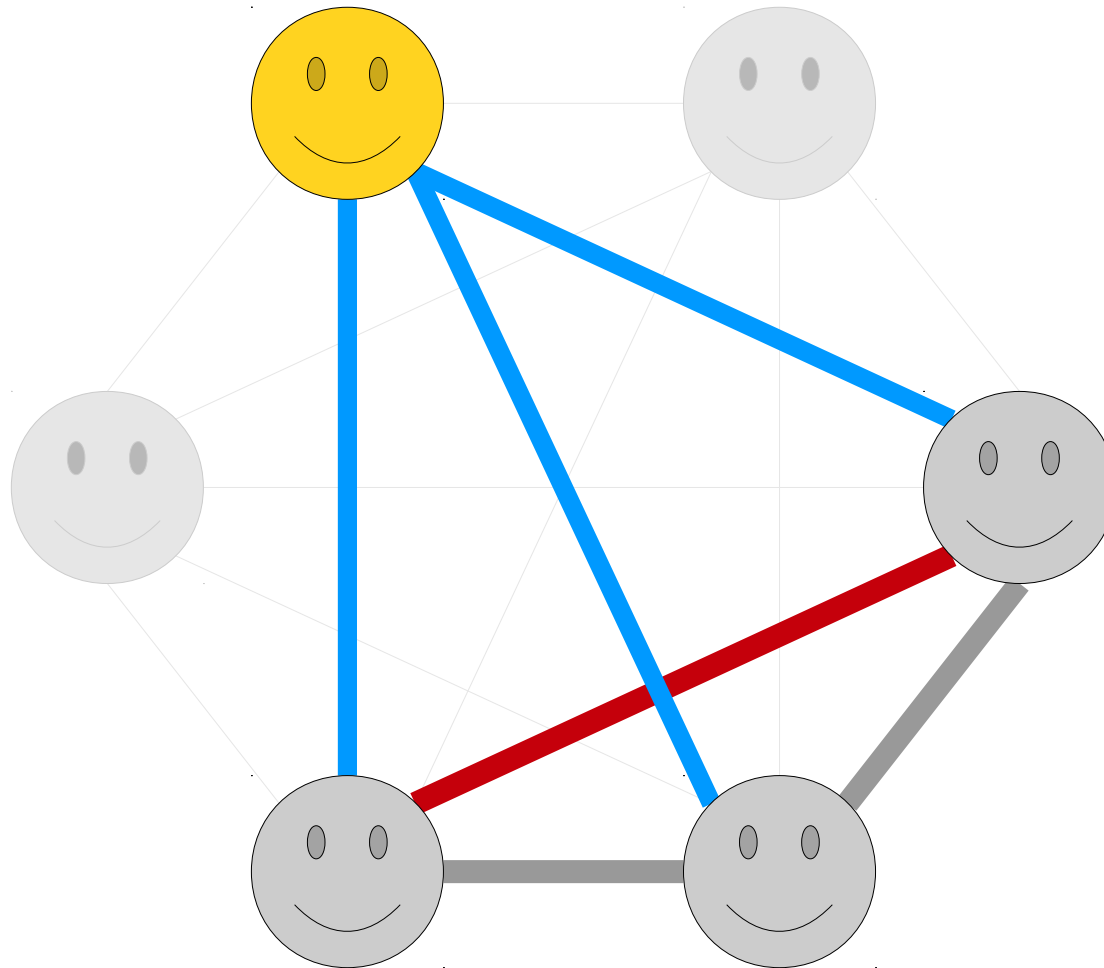


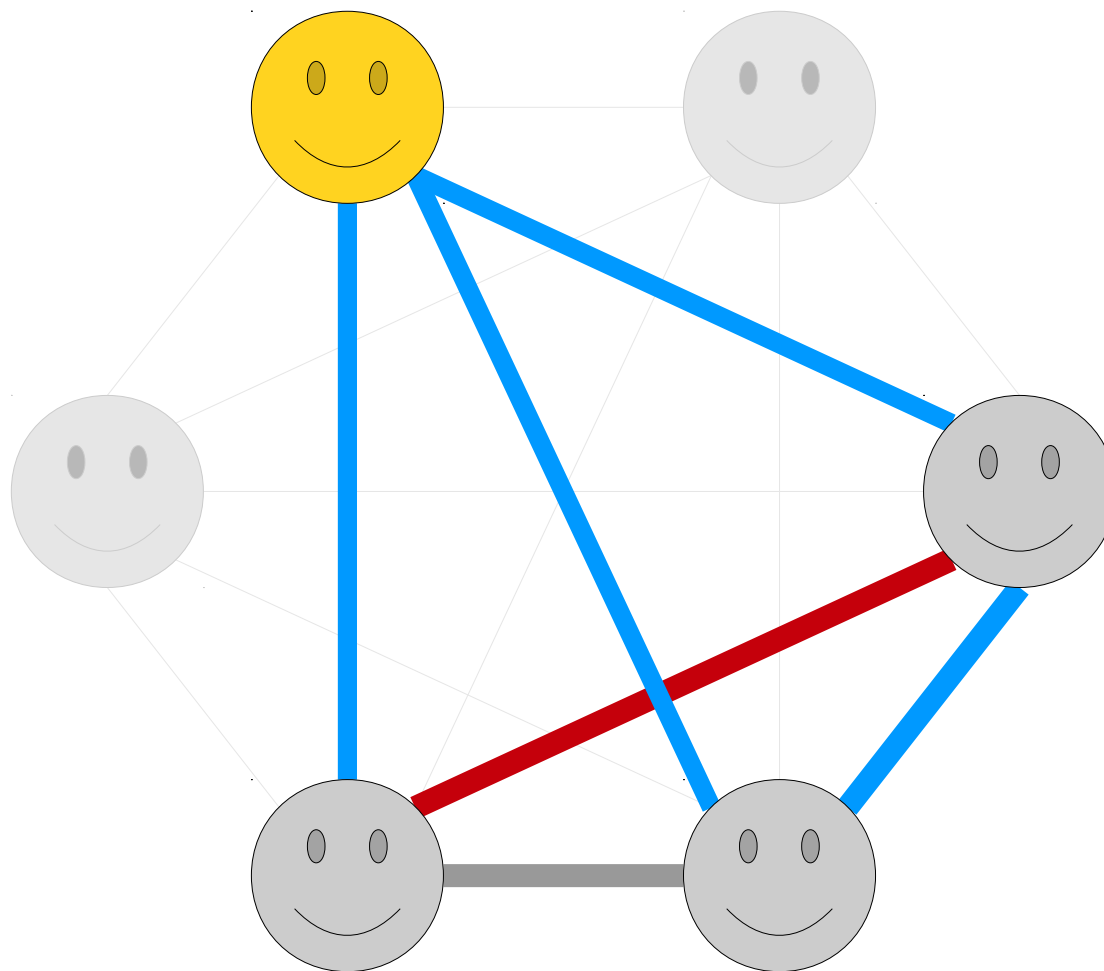
If we pick any node in the graph, that node will have at least  $\lceil 5/2 \rceil = 3$  edges of the same color incident to it.

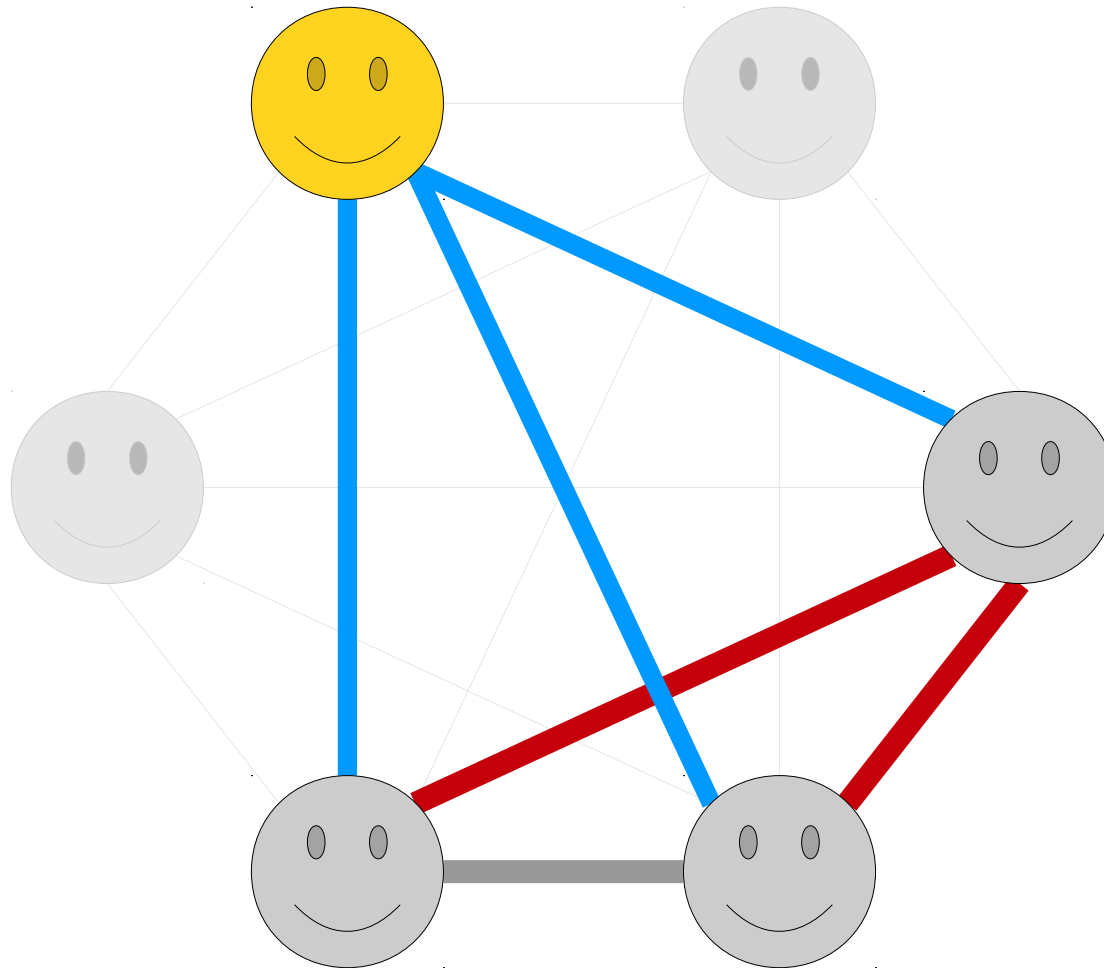


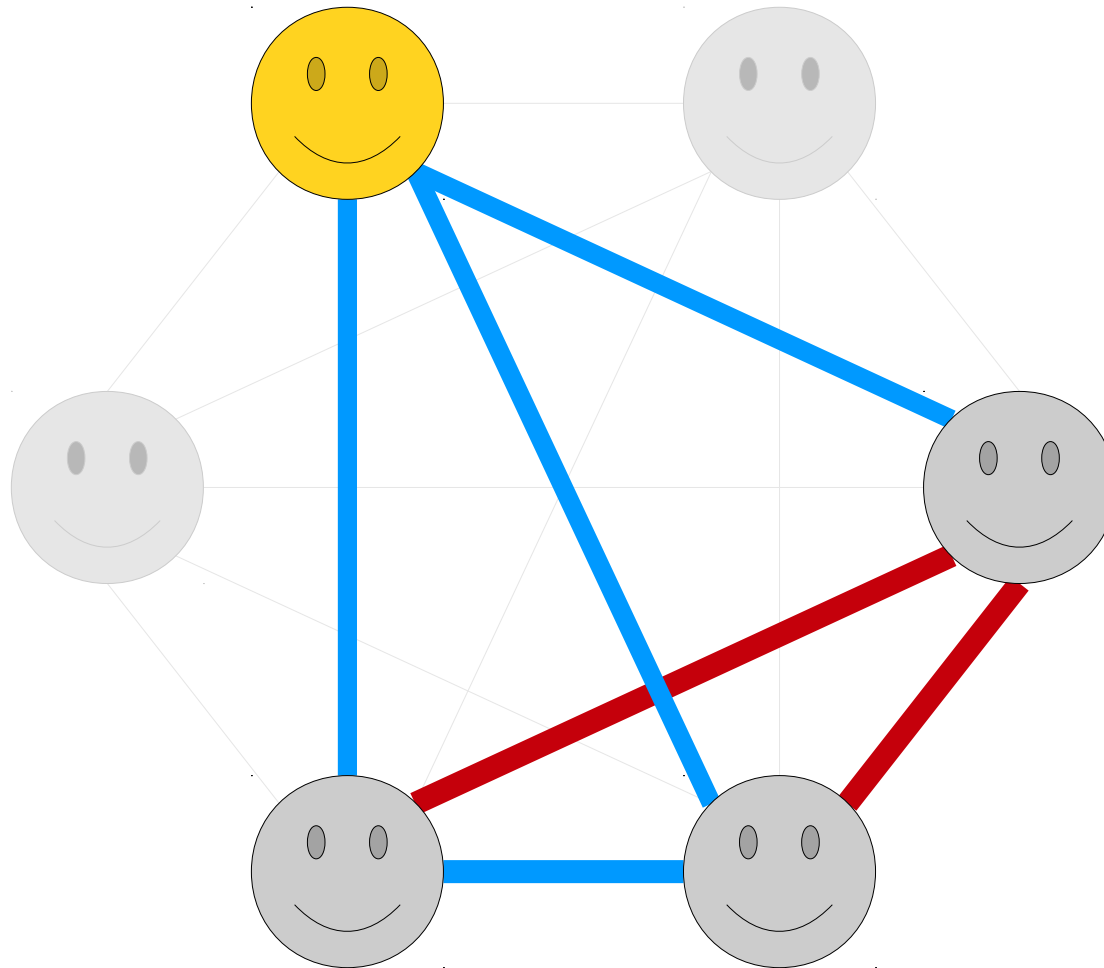


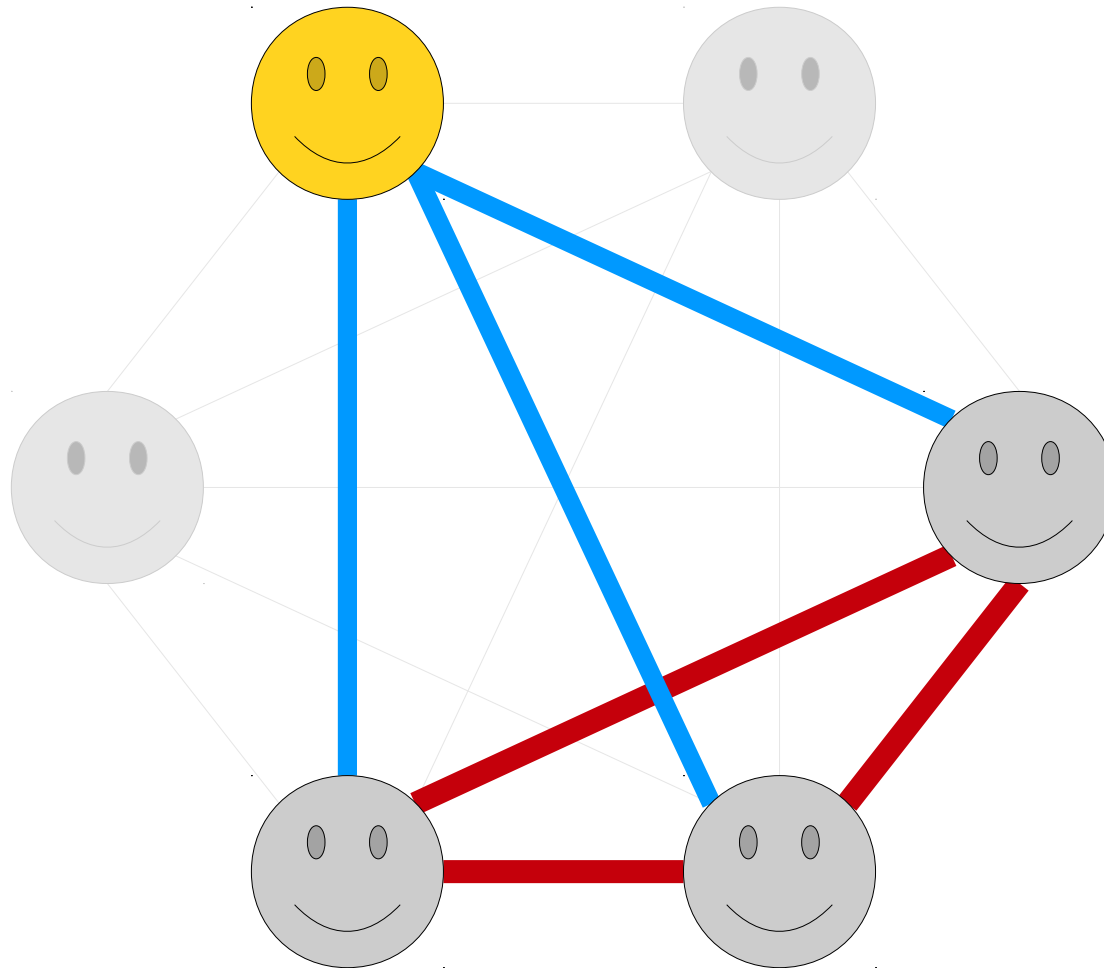












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The notation  $n!$  represents ***n factorial***, the product of all natural numbers between 1 and  $n$ , inclusive.

$$5! = 1 \times 2 \times 3 \times 4 \times 5.$$

The value  $3n!$  is read as  $3(n!)$ .

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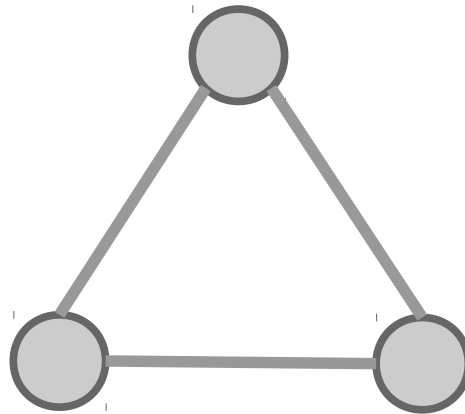
Based on this choice of  $P(n)$ , what are we trying to prove in the base case?

***Respond at [pollev.com/cs103](https://pollev.com/cs103)***

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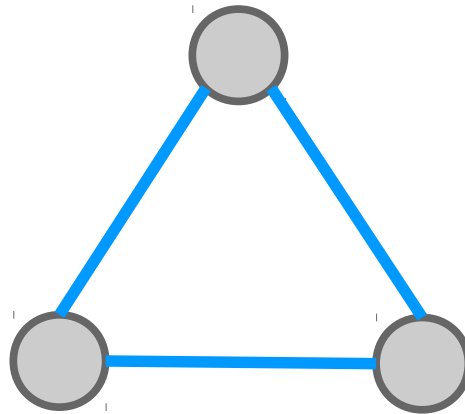
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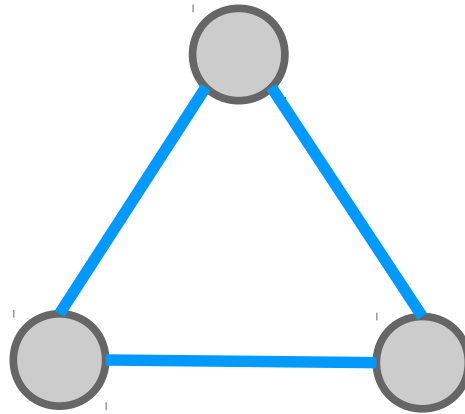
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Next, pick a natural number  $k \geq 1$  and assume  $P(k)$  is true

Based on this choice of  $P(n)$ , what are we assuming in the inductive step?

***Respond at [pollev.com/cs103](https://pollev.com/cs103)***

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To prove  $P(k+1)$ , what should we do?

- A) Pick a  $3k!$ -clique with edges colored using  $k$  colors, apply the inductive hypothesis, then add in nodes to create a larger  $3(k+1)!$ -clique.
- B) Pick a  $3(k+1)!$ -clique with edges colored using  $k+1$  colors, then discover a smaller  $3k!$ -clique within that larger clique to apply the inductive hypothesis to.
- C) Both options work.
- D) Neither option works.

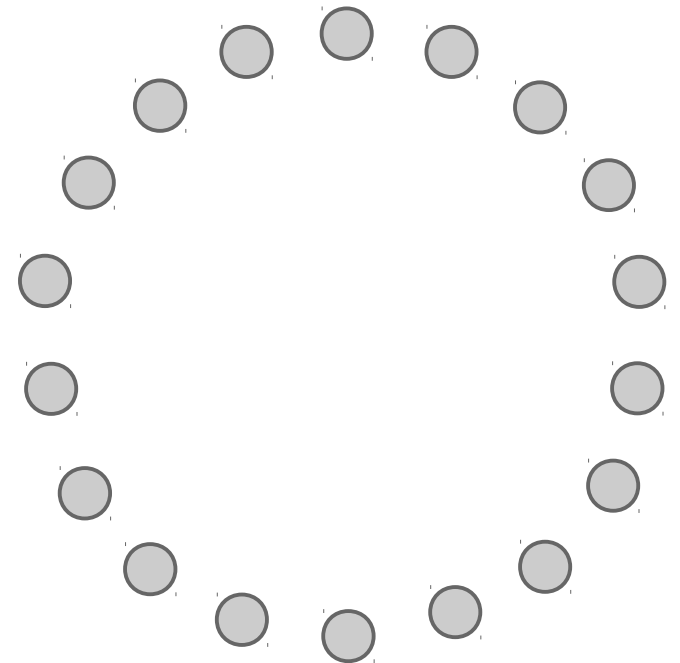
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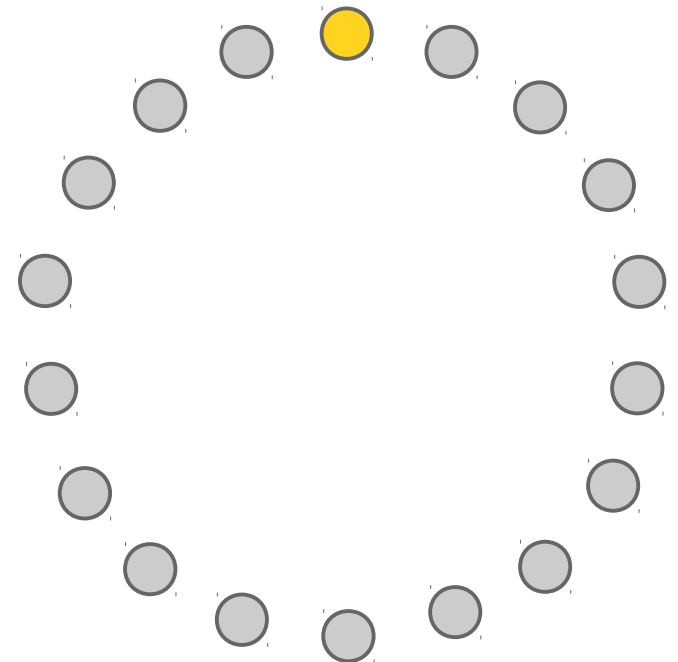
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How many edges have an endpoint at this node?  
How many possible edge colors do we have?

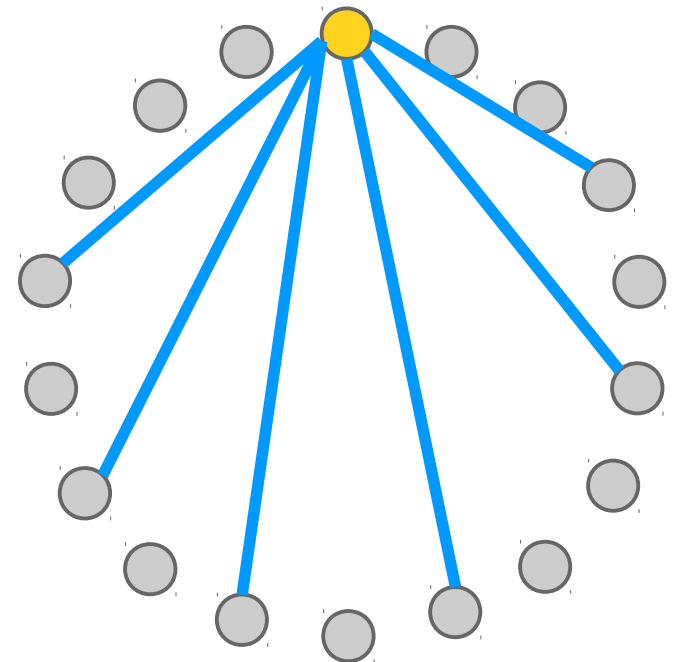


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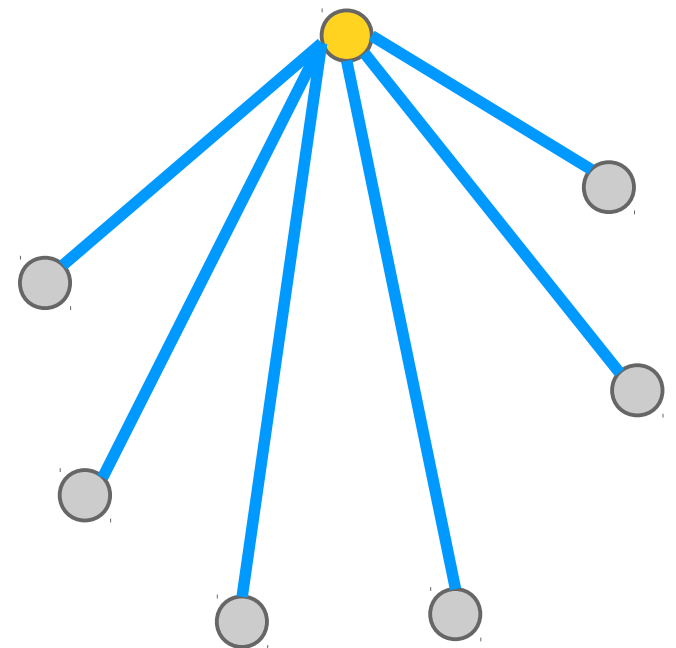
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Now let's look at these nodes that are adjacent to our chosen node via a blue edge. How do they relate to one another?



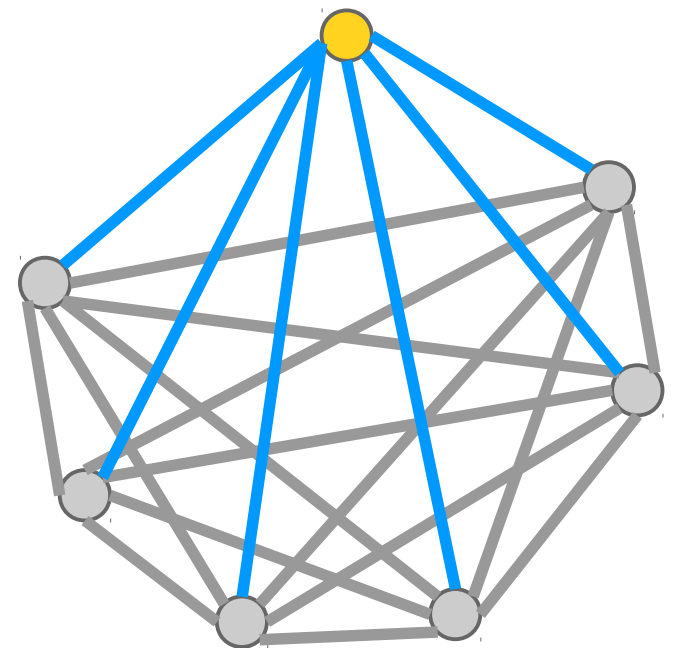
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Note: I’m coloring these edges in grey to indicate that we don’t know how these edges are colored, just that it’s some arbitrary coloring from the  $k+1$  possible colors.



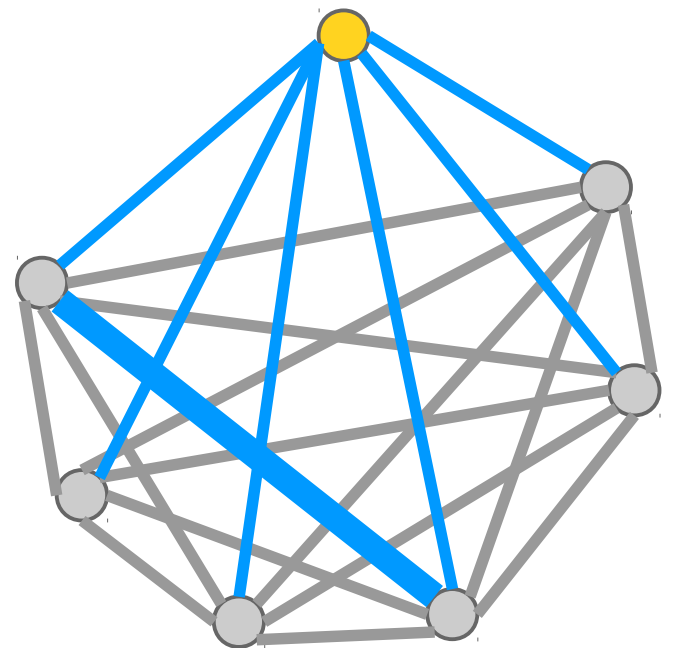
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Observation: if any one of these edges is blue, then we’ve found a blue triangle and we’re done.



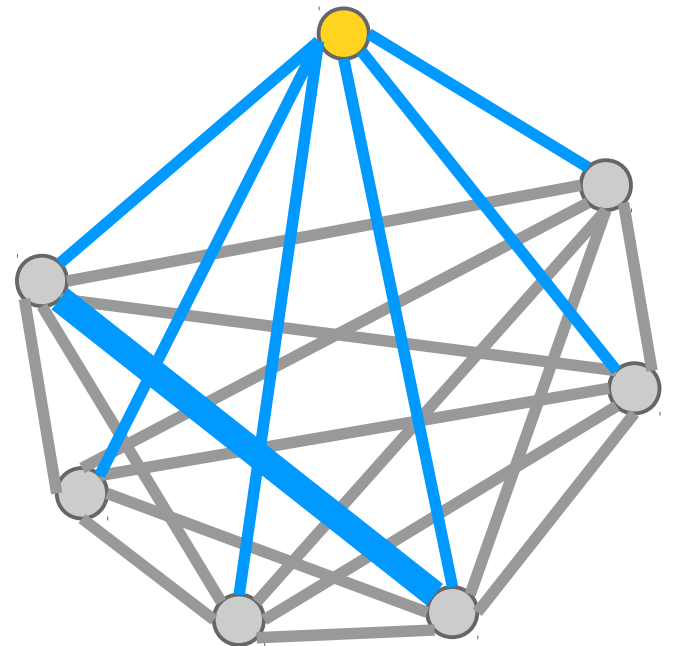
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So let's suppose that none of these edges are blue. What happens in that case?



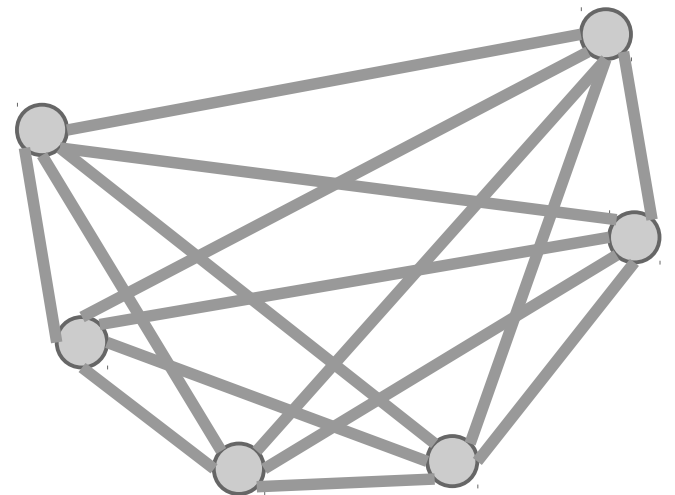
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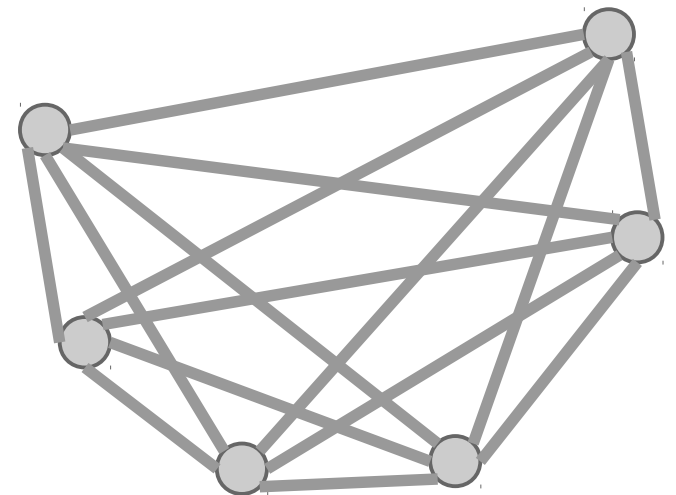
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Hey look, it's a clique! How many nodes does it have? How many possible colors are there for the edges? What does our inductive hypothesis say about cliques of that size?



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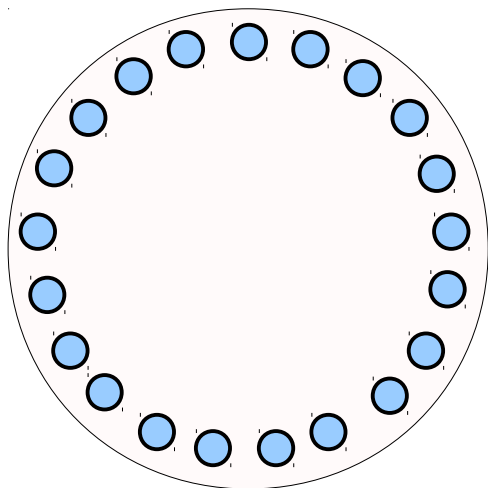
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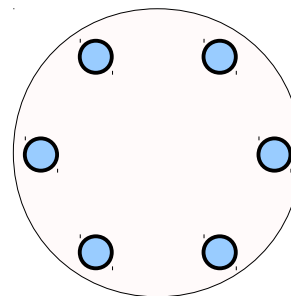
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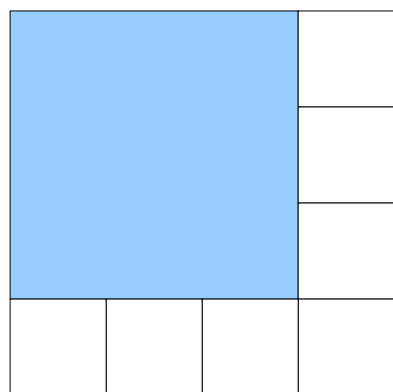
# An Observation



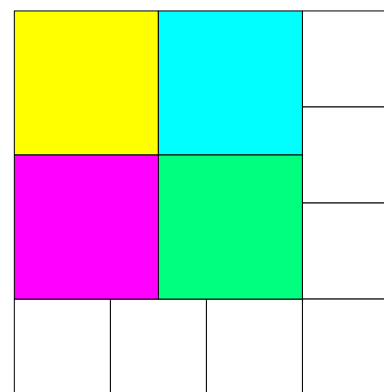
*Start with  
larger clique*



*Get to smaller  
clique*



*Start with  
fewer squares*



*Get to more  
squares*

# Following the Rules

- When working with square subdivisions, our predicate looked like this:

$P(n)$  is “**there exists** a way to subdivide a square into  $n$  squares.”

- When working with cliques, our predicate looked like this:

$P(n)$  is “**for any** coloring of a  $3n!$ -clique, there is a monochrome triangle.”

- With squares, the quantifier is  $\exists$ . With cliques, the first quantifier is  $\forall$ .
- This fundamentally changes the “feel” of induction.

# Build Up with $\exists$

- In the case of squares, in our inductive step, we prove

If

***there exists*** a subdivision into  $k$  squares,

then

***there exists*** a subdivision into  $k+3$  squares.

- Assuming the antecedent gives us a concrete subdivision into  $k$  squares.
- Proving the consequent means finding some way to subdivide in to  $k+3$  squares.
- The inductive step goal is to “***build up:***” start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

# Build Down with $\forall$

- In the case of cliques, in our inductive step, we prove

If

**for all** colorings of a  $3k!$ -clique, there's a mono. tri.

then

**for all** colorings of a  $3(k+1)!$ -clique, there's a mono. tri.

- Assuming the antecedent means once we find a  $k$ -colored  $3k!$ -clique, we get a monochrome triangle.
- Proving the consequent means picking an arbitrary coloring of a  $3(k+1)!$ -clique, then trying to find a triangle in it.
- The inductive step goal is to “**build down:**” start with a larger clique, then find a way to turn it into a smaller clique.

# More on Ramsey Triangles

- We've proved that  $3n!$  nodes is enough to get a triangle with  $n \geq 1$  colors on the edges.
- For  $n = 3$ , this says we need 18 nodes, on PS4 you'll prove that you can do this with just 17 nodes.
- For  $n = 4$ , this says we need 72 nodes. We know that 50 nodes is too few and 66 nodes is enough. The actual answer is therefore somewhere between 51 and 66.
- **Open problem:** Find the exact minimum number of nodes needed to get a monochrome triangle with  $n \geq 4$  colors.
- **Challenge problem:** Show that  $\lceil e \cdot n! \rceil$  nodes is always sufficient to get a monochrome triangle with  $n \geq 1$  colors. *(This is hard but doable if you know the material from CS103, plus the Taylor series for  $e$ .)*

Let's take a quick break!

**Time-Out for Announcements!**

# Problem Sets

- Problem Set Three was due today at 4:00PM.
- Problem Set Four goes out today. It's due next Friday at 4:00PM.
  - Because this coincides with the day of the midterm, we are implementing the following policy:
    - On-time submissions will receive a small bonus.
    - There is a penalty-free 48 hour grace period to submit until Sunday at 4:00PM.
    - This policy applies for this assignment only.

# Midterm Exam Logistics

- Our midterm exam will be on Friday, July 28<sup>th</sup> from 4:30 – 7:30 PM in Shriram 104 (our normal lecture room).
- You're responsible for lectures up to the end of week 3 and topics from PS1 – PS3. Later lectures and problem sets won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is open-book, open-note, and closed-other-humans/AI.

# Midterm Accommodations

- This is your *last call* for midterm accommodations:
  - If you have OAE accommodations, you should have received an email from us with exam time and location.
  - If you have a midterm conflict, you should have received an email from us with instructions on how you will be taking the exam.
  - If you fall into either of these categories but have not heard from us, email the course staff *ASAP* at [cs103-sum2223-staff@lists.stanford.edu](mailto:cs103-sum2223-staff@lists.stanford.edu).

# Midterm Review Session

- Your wonderful TAs Benson and Annika are organizing a midterm review session.
- Next Tuesday 7/25 from 5:00 – 6:00PM PT, over Zoom.
- Link to join is on our office hours calendar.
- Please join live to ask questions! The session will also be recorded for asynchronous viewing.

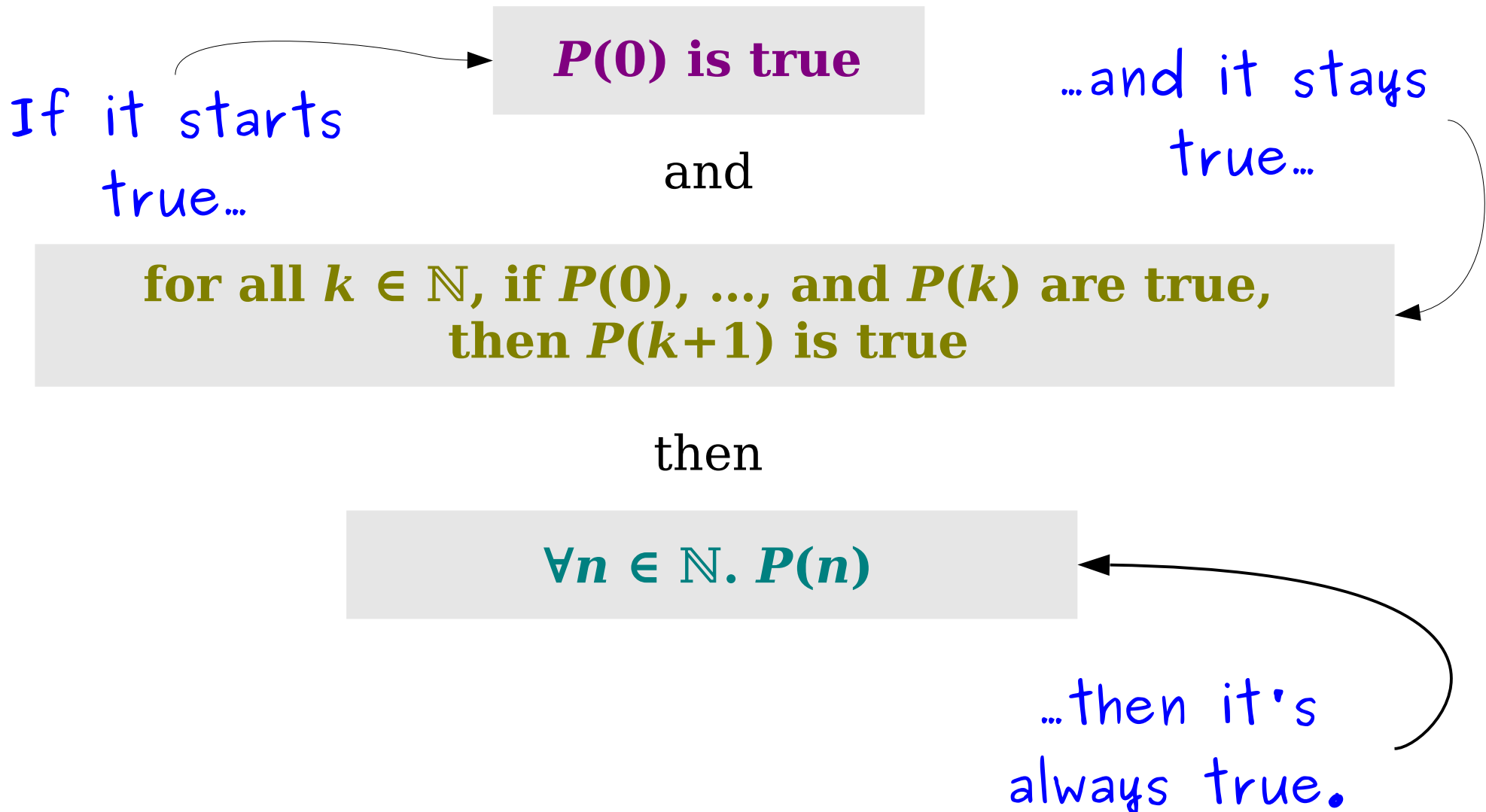
# Preparing for the Exam

- Review your assignment feedback and the solutions and make sure you understand our comments.
- Practice Midterm 1 - slightly easier than our exam.
- Practice Midterm 2 - approximately the same difficulty as our exam.
- 30 Extra Practice Problems across all topics.
- Please do ***not*** read the solutions to a problem until you have worked through it.

Let's get back to CS103!

# Complete Induction

Let  $P$  be some predicate. The **principle of complete induction** states that if



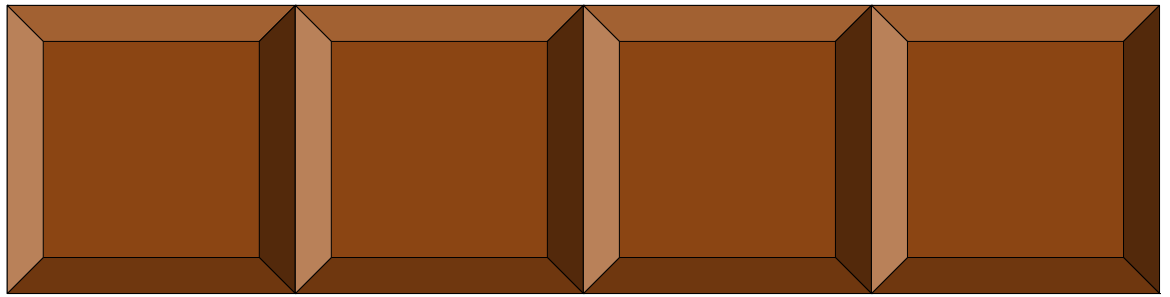
# Mathematical Induction

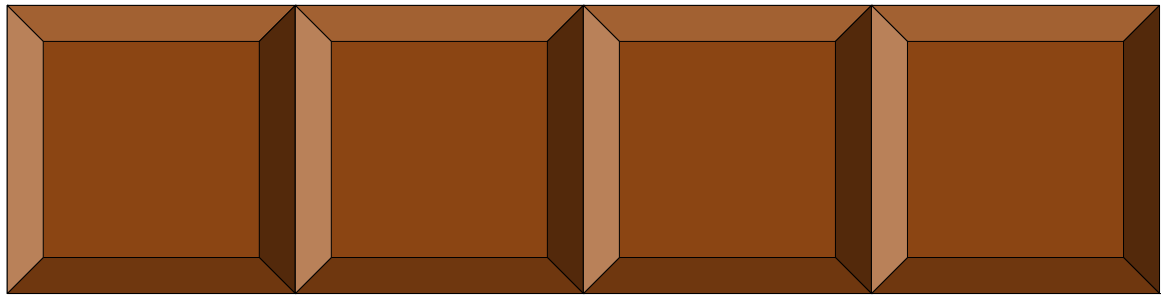
- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate  $P(n)$  to prove by induction on  $n$ .
  - Choose and prove a base case (probably, but not always,  $P(0)$ ).
  - Pick an arbitrary  $k \in \mathbb{N}$  and assume that  $P(k)$  is true.
  - Prove  $P(k+1)$ .
  - Conclude that  $P(n)$  holds for all  $n \in \mathbb{N}$ .

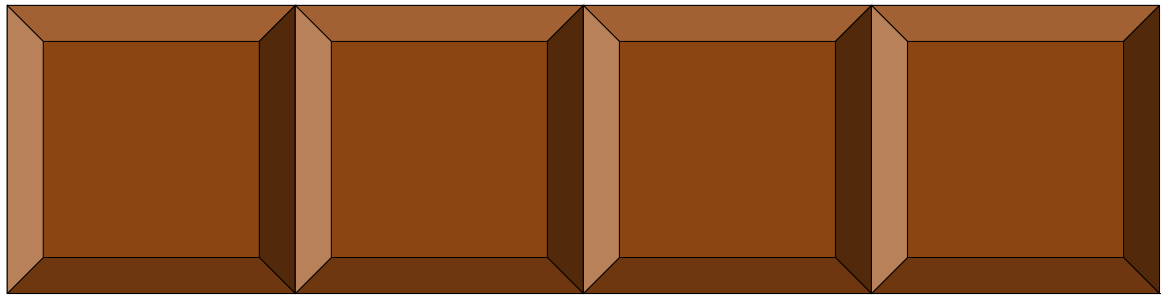
# Complete Induction

- You can write proofs using the principle of **complete** induction as follows:
  - Define some predicate  $P(n)$  to prove by induction on  $n$ .
  - Choose and prove a base case (probably, but not always,  $P(0)$ ).
  - Pick an arbitrary  $k \in \mathbb{N}$  and assume that  **$P(0), P(1), P(2), \dots,$  and  $P(k)$**  are all true.
  - Prove  $P(k+1)$ .
  - Conclude that  $P(n)$  holds for all  $n \in \mathbb{N}$ .

An Example: *Eating a Chocolate Bar*

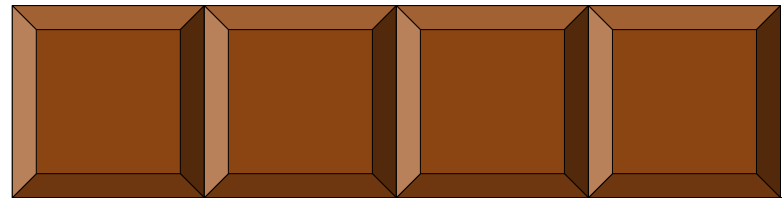


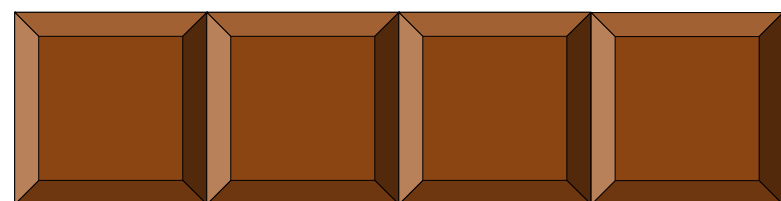
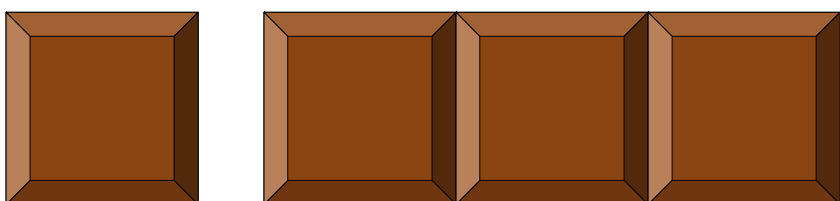
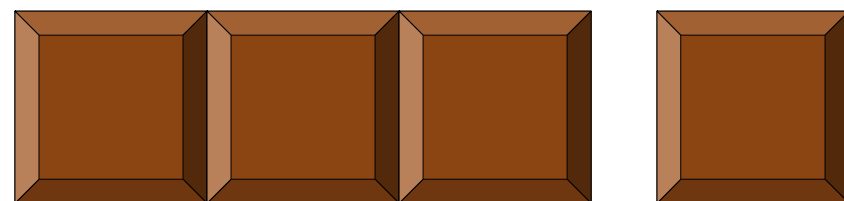
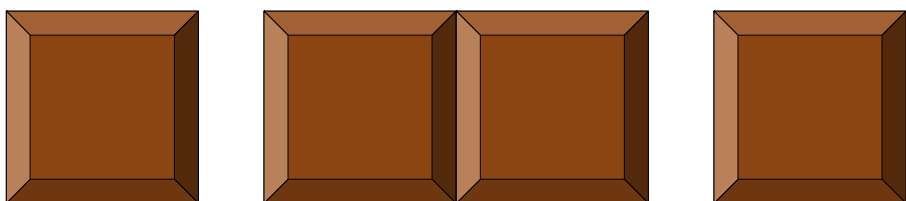
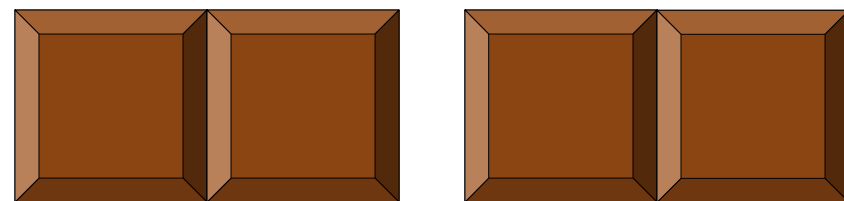
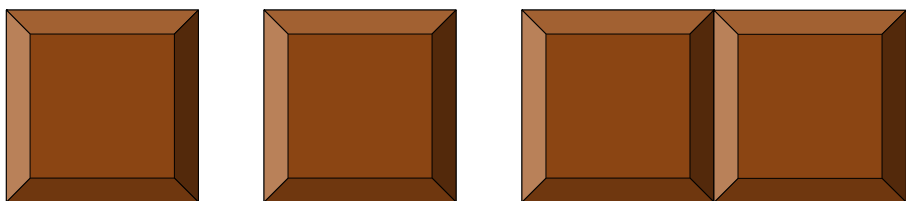
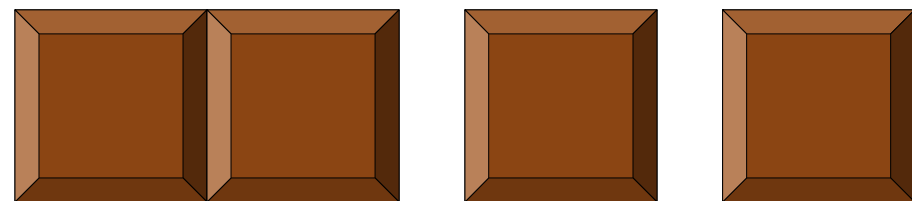
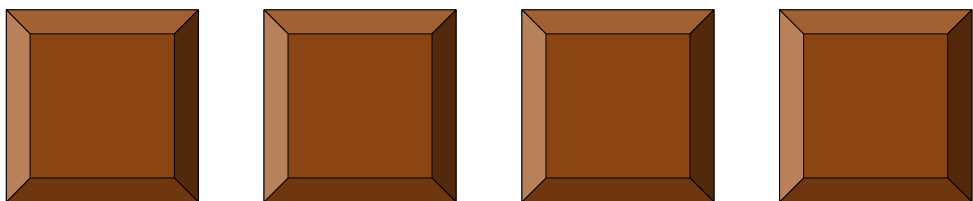




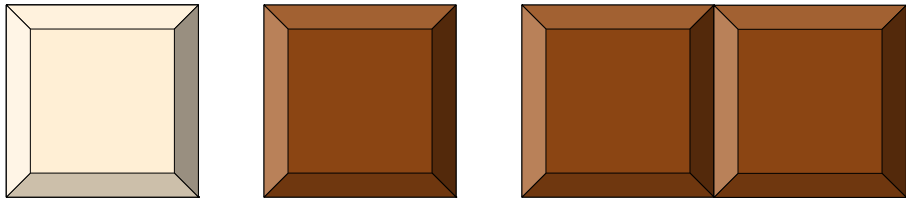
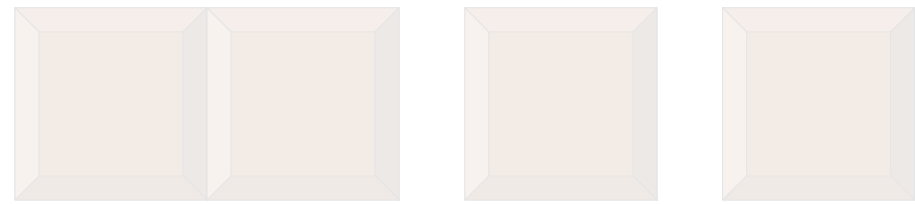
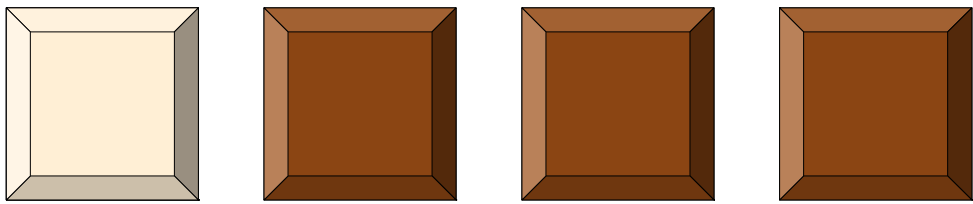
# Eating a Chocolate Bar

- You have a  $1 \times n$  chocolate bar subdivided into  $1 \times 1$  squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a...
  - $1 \times 1$  chocolate bar?
  - $1 \times 2$  chocolate bar?
  - $1 \times 3$  chocolate bar?
  - $1 \times 4$  chocolate bar?

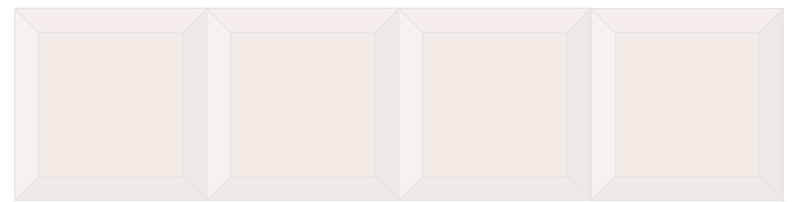
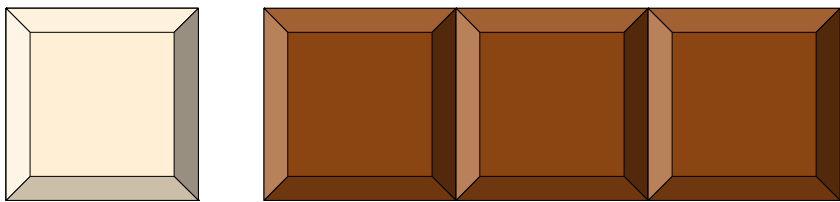
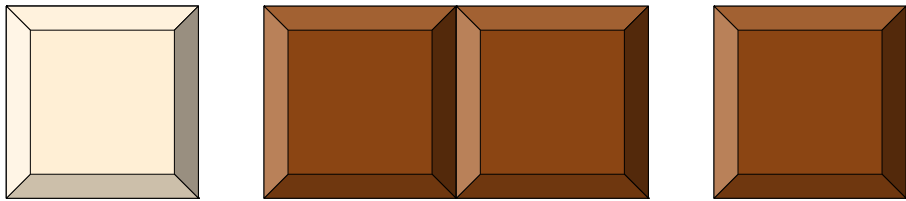




There are eight ways to eat a  $1 \times 4$  chocolate bar.



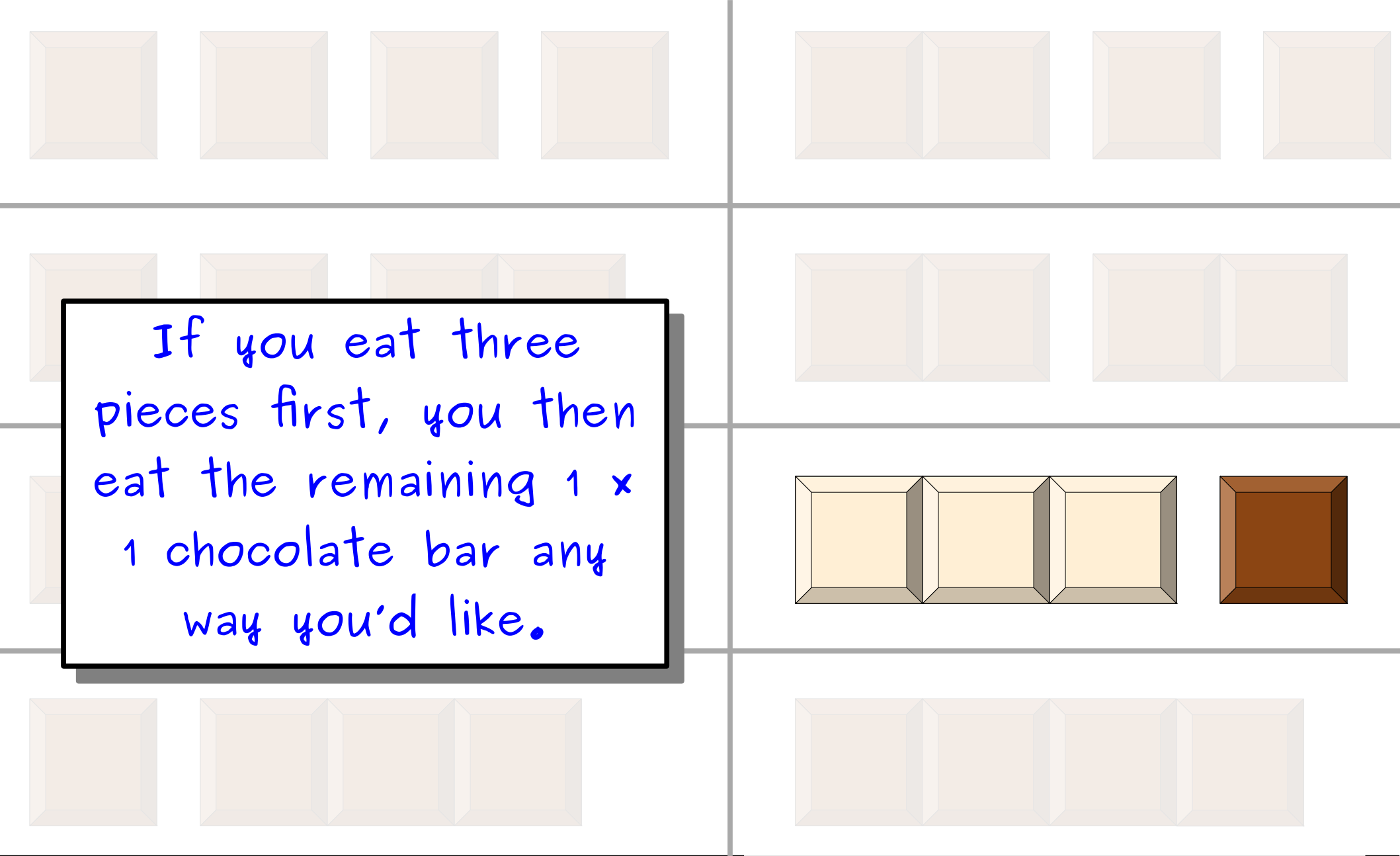
If you eat one piece first, you then eat the remaining  $1 \times 3$  chocolate bar any way you'd like.



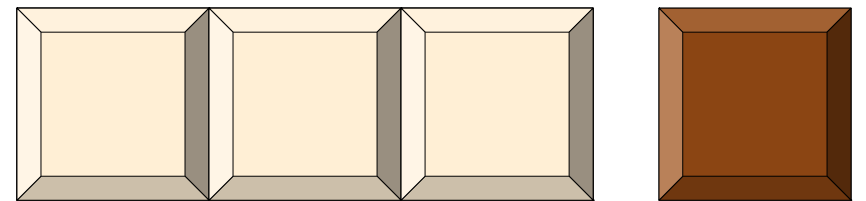
There are eight ways to eat a  $1 \times 4$  chocolate bar.

If you eat two pieces first, you then eat the remaining  $1 \times 2$  chocolate bar any way you'd like.

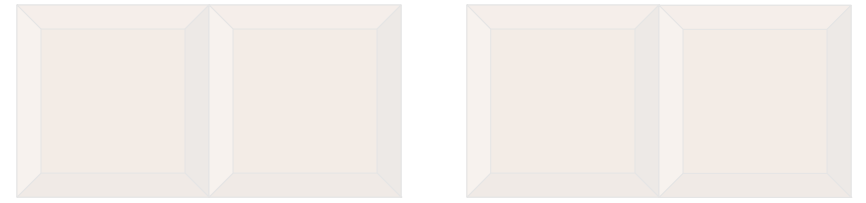
There are eight ways to eat a  $1 \times 4$  chocolate bar.



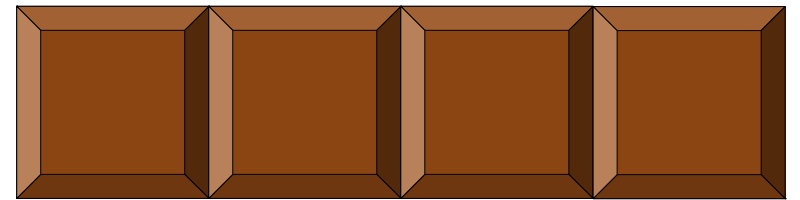
If you eat three pieces first, you then eat the remaining  $1 \times 1$  chocolate bar any way you'd like.



There are eight ways to eat a  $1 \times 4$  chocolate bar.



Or you could eat the whole chocolate bar at once. Ah, gluttony.



There are eight ways to eat a  $1 \times 4$  chocolate bar.

# Eating a Chocolate Bar

- There's...
  - 1 way to eat a  $1 \times 1$  chocolate bar,
  - 2 ways to eat a  $1 \times 2$  chocolate bar,
  - 4 ways to eat a  $1 \times 3$  chocolate bar, and
  - 8 ways to eat a  $1 \times 4$  chocolate bar.
- ***Our guess:*** There are  $2^{n-1}$  ways to eat a  $1 \times n$  chocolate bar for any natural number  $n \geq 1$ .
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size  $k$ , then eating the remaining  $n - k$  pieces however we'd like.
- Let's formalize this!

**Theorem:** For any natural number  $n \geq 1$ , there are exactly  $2^{n-1}$  ways to eat a  $1 \times n$  chocolate bar from left to right.

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Summing up this first option, plus all choices of  $r$  for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2^1 + 2^0$$

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**Proof:** Let  $P(n)$  be the statement “there are exactly  $2^{n-1}$  ways to eat a  $1 \times n$  chocolate bar from left to right.” We will prove by induction that  $P(n)$  holds for all natural numbers  $n \geq 1$ , from which the theorem follows.

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For our inductive step, assume for some arbitrary natural number  $k \geq 1$  that  $P(1)$ , ..., and  $P(k)$  are true. We need to show  $P(k+1)$  is true, that there are exactly  $2^k$  ways to eat a  $1 \times (k+1)$  chocolate bar.

There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size  $r$  for some  $1 \leq r \leq k$ , leaving a chocolate bar of size  $k+1-r$ , then eat that chocolate bar from left to right. Since  $1 \leq r \leq k$ , we know that  $1 \leq k+1-r \leq k$ , so by our inductive hypothesis there are  $2^{k-r}$  ways to eat the remainder.

Summing up this first option, plus all choices of  $r$  for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2^1 + 2^0 = 1 + 2^k - 1 = 2^k.$$

Thus  $P(k+1)$  holds, completing the induction.

**Theorem:** For any natural number  $n \geq 1$ , there are exactly  $2^{n-1}$  ways to eat a  $1 \times n$  chocolate bar from left to right.

**Proof:** Let  $P(n)$  be the statement “there are exactly  $2^{n-1}$  ways to eat a  $1 \times n$  chocolate bar from left to right.” We will prove by induction that  $P(n)$  holds for all natural numbers  $n \geq 1$ , from which the theorem follows.

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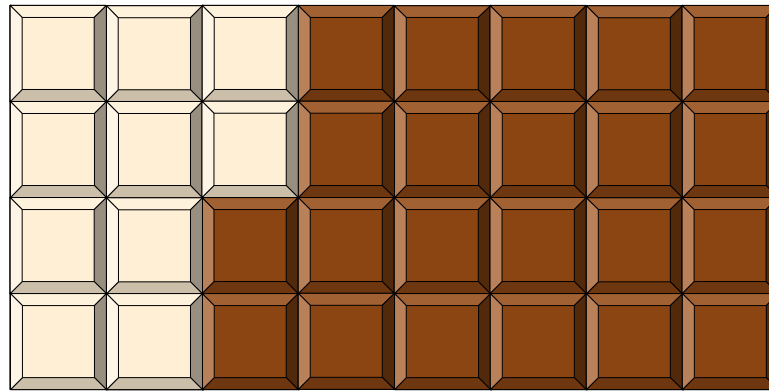
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Thus  $P(k+1)$  holds, completing the induction. ■

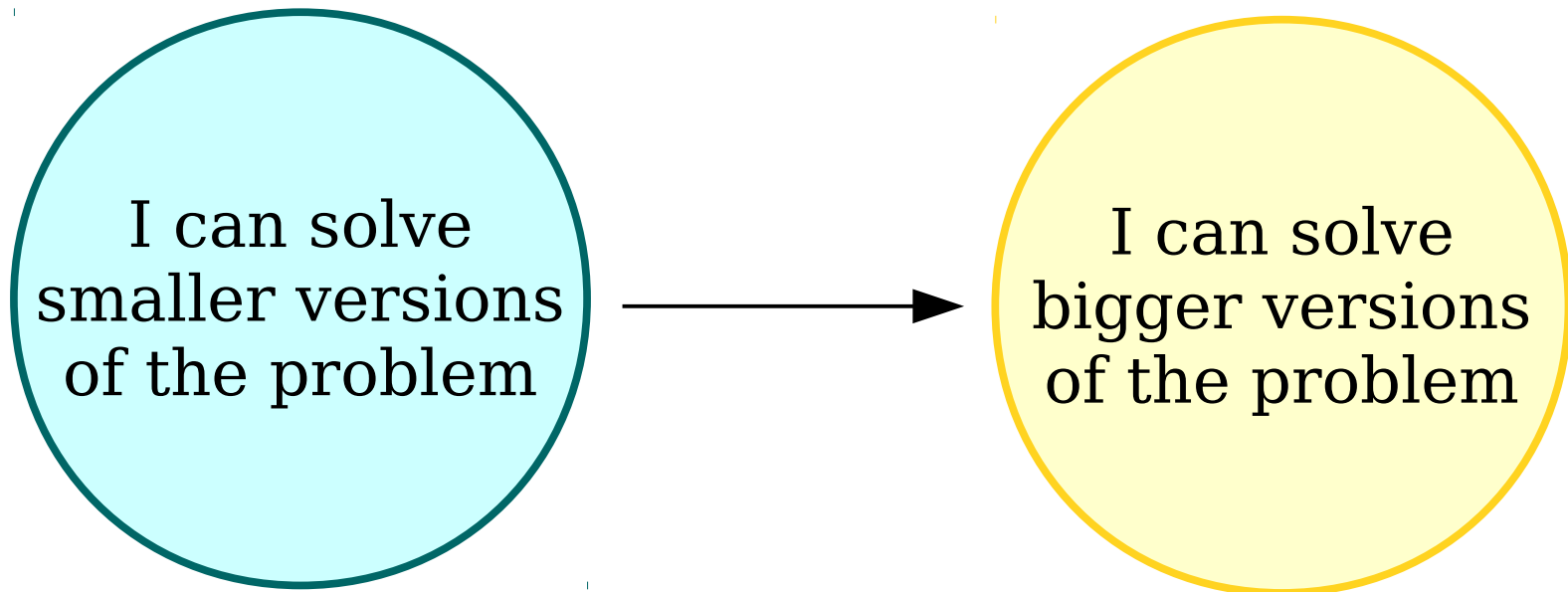
# More on Chocolate Bars

- Imagine you have an  $m \times n$  chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?

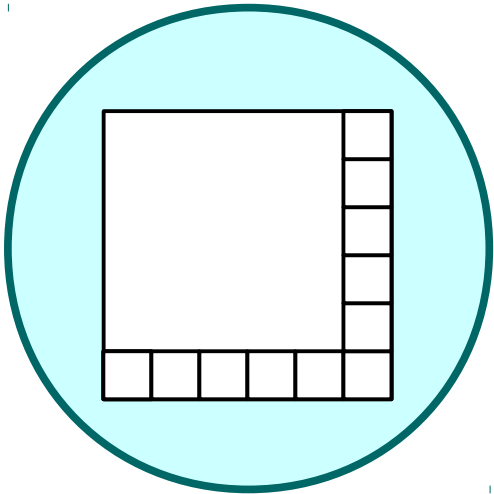


- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as  $m$  and  $n$  tend toward infinity.

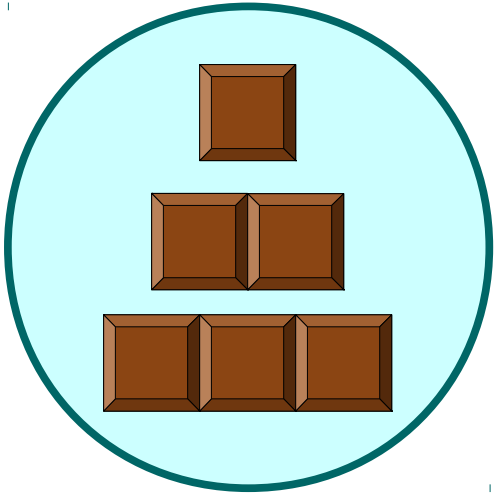
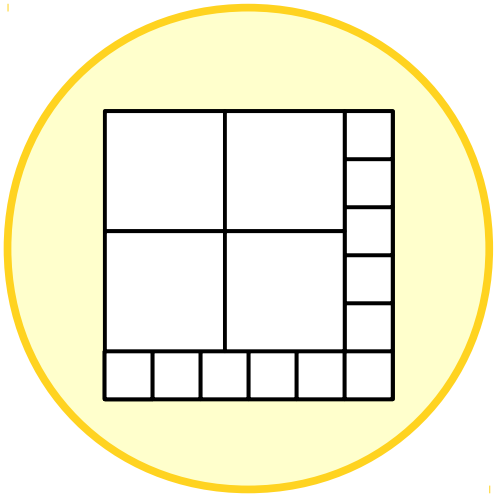
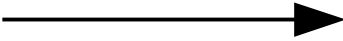
# Induction vs. Complete Induction



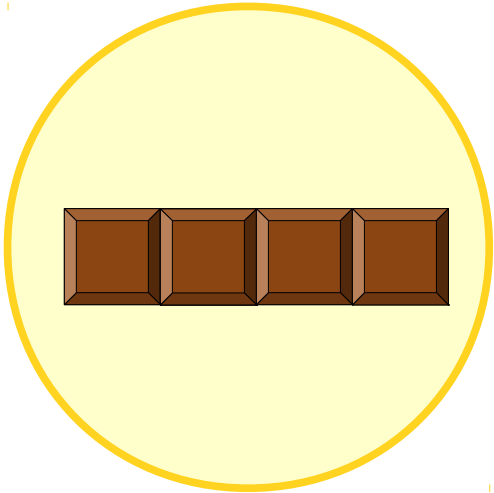
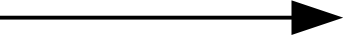
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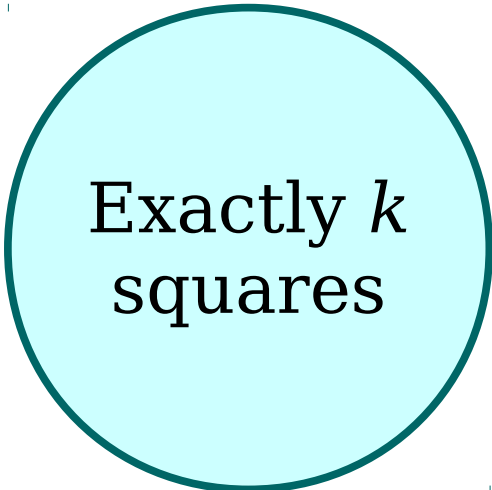
Regular Induction



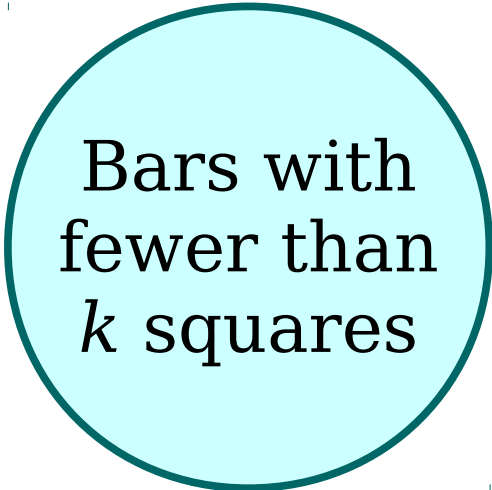
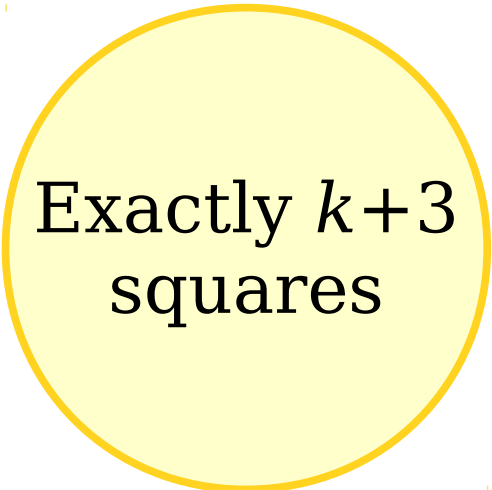
Complete Induction



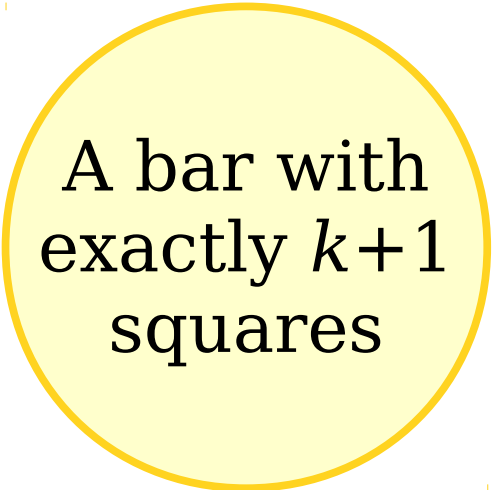
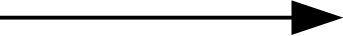
# Induction vs. Complete Induction



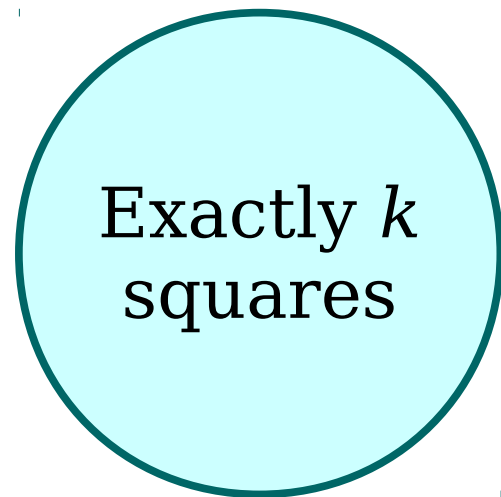
Regular Induction



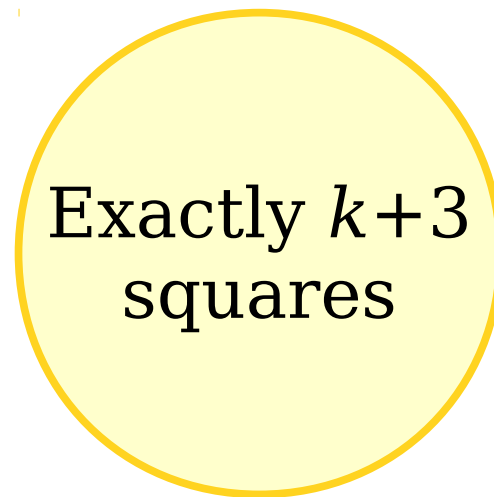
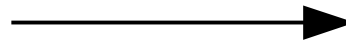
Complete Induction



# Induction vs. Complete Induction

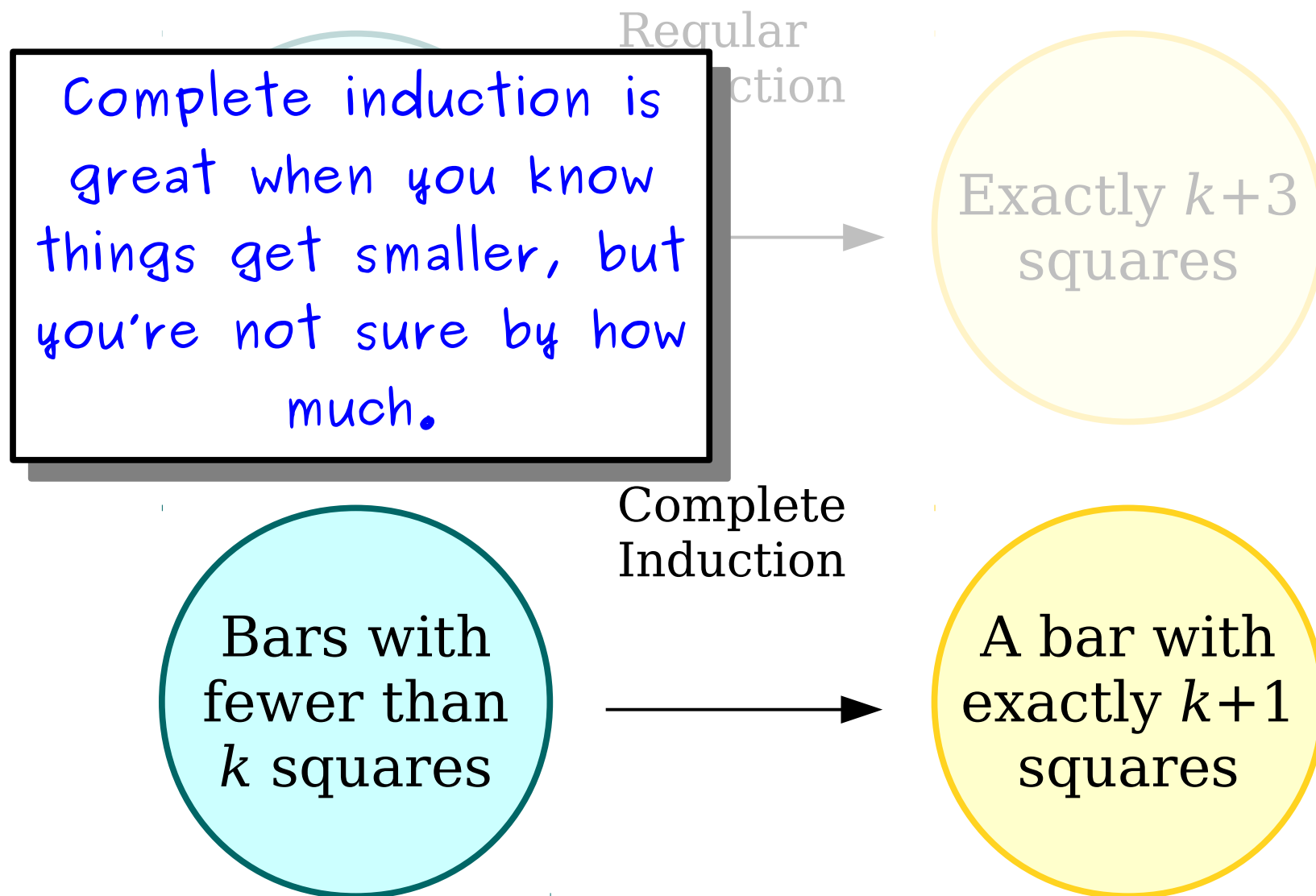


Regular  
Induction



Regular induction is great when you know exactly how much smaller your "smaller" problem instance is.

# Induction vs. Complete Induction



***An Important Milestone***

# Recap: *Discrete Mathematics*

- The past five weeks have focused exclusively on discrete mathematics:

Induction

Functions

Graphs

The Pigeonhole Principle

Formal Proofs

Mathematical Logic

Set Theory

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

# Three Questions

- What is something you know now that, at the start of the quarter, you knew you didn't know?
- What is something you know now that, at the start of the quarter, you didn't know that you didn't know?
- What is something you don't know that, at the start of the quarter, you didn't know that you didn't know?

# Next Up: *Computability Theory*

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

# Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.